MIXED TORIC RESIDUES AND TROPICAL DEGENERATIONS

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0. Introduction

This paper is a follow-up to our paper [15], where we prove a conjecture of Batyrev and Materov, the Toric Residue Mirror Conjecture (TRMC). Here we extend our results, and show that they imply a generalization of this conjecture, the Mixed Toric Residue Mirror Conjecture (MTRMC), which is also due to Batyrev and Materov [3].

Roughly, these conjectures state that the generating function of certain intersection numbers of a sequence of toric varieties converges to a rational function, which can be obtained as a finite residue sum on a single toric variety. We first recall the TRMC in some detail. We start with an integral convex polytope $\Pi^{\mathfrak{B}}$ in a d-dimensional real vector space $\mathfrak{t}$ endowed with a lattice of full rank $\mathfrak{t}_{\mathbb{Z}}$; we assume that the polytope contains the origin in its interior. Let the sequence $\mathfrak{B} = [\beta_1, \beta_2, \ldots, \beta_n]$ be the set of vertices of this polytope, ordered in an arbitrary fashion. One can associate a d-dimensional polarized toric variety $(V^{\mathfrak{B}}, L^{\mathfrak{B}})$ to this data in the standard fashion [15].

There is another way to obtain toric varieties from this data, which generalizes the mirror duality of polytopes introduced by Batyrev [1]. Consider the sequence $\mathfrak{A} = [\alpha_1, \alpha_2, \ldots, \alpha_n]$, which is the Gale dual of $\mathfrak{B}$ (cf. §1.3 for the construction). This is a sequence of integral vectors in the dual $\mathfrak{a}^*$ of a certain $r = n - d$-dimensional vector space $\mathfrak{a}$, which is also endowed with a lattice of full rank: $\mathfrak{a}_{\mathbb{Z}}$; in this setup the sequence $\mathfrak{A}$ spans a strictly convex cone Cone($\mathfrak{A}$). The simplicial cones generated by $\mathfrak{A}$ divide Cone($\mathfrak{A}$) into open chambers. Each chamber corresponds to a d-dimensional orbifold toric variety $V_{\mathfrak{A}}(\mathfrak{c})$ (cf. [8]). An integral element $\alpha \in \mathfrak{a}^*$ specifies an orbi-line-bundle L_α over this variety; denote the first chern class of L_α by $\chi(\alpha) \in H^2(V_{\mathfrak{A}}(\mathfrak{c}), \mathbb{Q})$. For the purposes of this introduction we assume that this correspondence induces the linear isomorphisms

$$\mathfrak{a}^* \cong H^2(V_{\mathfrak{A}}(\mathfrak{c}), \mathbb{R}) \quad \text{and} \quad \mathfrak{a} \cong H_2(V_{\mathfrak{A}}(\mathfrak{c}), \mathbb{R})$$

Now pick a chamber $\mathfrak{c}$ which contains the vector $\kappa = \sum_{i=1}^n \alpha_i$ in its closure: $\kappa \in \bar{\mathfrak{c}}$. To each element $\lambda \in \mathfrak{a}_{\mathbb{Z}}$, one can associate a moduli space MP_λ, the so-called Morrison-Plesser space (cf. [12]), which is a compactification of the space of those maps from the projective line to the variety $V_{\mathfrak{A}}(\mathfrak{c})$ under which the image of the fundamental class is λ:

$$\{m : \mathbb{P}^1 \to V_{\mathfrak{A}}(\mathfrak{c});\ m_*([\mathbb{P}^1]) = \lambda\}.$$

The varieties MP_λ are toric, and such that, again, to each integral element $\alpha \in \mathfrak{a}^*$ one can associate a line bundle L_α on MP_λ; again we denote the corresponding chern class in $H^2(\mathrm{MP}_\lambda)$ by $\chi(\alpha)$. The space MP_λ is defined to be empty, unless $\langle\alpha,\lambda\rangle \geq 0$ for every $\alpha \in \mathfrak{c}$. The set of vectors satisfying this condition forms a cone in $\mathfrak{a}$, which we denote by $\bar{\mathfrak{c}}^\perp$; this cone is called the *polar* cone of $\mathfrak{c}$.

The construction also provides a Poincare dual class

$$K_\lambda \in H^{2(\dim \mathrm{MP}_\lambda - d)}(\mathrm{MP}_\lambda, \mathbb{Q})$$

to the subspace of MP_λ of those maps which land in a generic zero-section Y of the line bundle L_κ. When $V_{\mathfrak{A}}(\mathfrak{c})$ is smooth, then Y is a Calabi-Yau manifold.

To probe the class K_λ, we fix a homogeneous polynomial $P(x_1,\dots,x_n)$ of degree d in n variables, and consider the intersection numbers

$$\int_{\mathrm{MP}_\lambda} P(\chi(\alpha_1),\dots,\chi(\alpha_n))K_\lambda,$$

which are to be interpreted as analogs of numbers of rational curves in Y subject to certain conditions specified by the polynomial P.

Now let $z_1,\dots,z_n \in \mathbb{C}^*$, and form the Laurent series

$$\sum_{\lambda\in\mathfrak{a}_\mathbb{Z}} \int_{\mathrm{MP}_\lambda} P(\chi(\alpha_1),\dots,\chi(\alpha_n))K_\lambda \prod_{i=1}^n z_i^{\langle\alpha_i,\lambda\rangle}. \tag{0.1}$$

The statement of the conjecture is that this series converges in a certain domain of $(\mathbb{C}^*)^n$ depending on the chamber $\mathfrak{c}$, and the limit is a rational function of the parameters $z_1,\dots,z_n$, which is given by a *toric residue* on the polarized toric variety $(V^{\mathfrak{B}}, L^{\mathfrak{B}})$. This limiting function does not depend on which chamber $\mathfrak{c}$ with $\kappa \in \bar{\mathfrak{c}}$ we picked.

Toric residues for polarized toric varieties were introduced by Cox [7]. Under appropriate conditions, a polarized toric variety (V, L) of dimension d, and a generic $(d+1)$-tuple of homogeneous functions $\mathbf{f}(z) = (f_0, f_1, \dots, f_d)$ in $H^0(L, V)$ defines a residue functional on the homogeneous ring

$$Q \longmapsto \mathrm{TorRes}_{\mathbf{f}(z)}Q.$$

This functional vanishes on the ideal generated by $\mathbf{f}(z)$, and depends rationally on $\mathbf{f}(z)$ and Q. In our case, by definition, the space of sections $H^0(L^{\mathfrak{B}}, V^{\mathfrak{B}})$ of the polarizing line bundle $L^{\mathfrak{B}}$ is spanned by $\{e_\beta;\ \beta \in \Pi^{\mathfrak{B}} \cap \mathfrak{t}_\mathbb{Z}\}$, where e_β is a character of the torus $\mathfrak{t}^*/\mathfrak{t}^*_\mathbb{Z}$. In particular, the section $f_0 = 1 - \sum_{i=1}^n z_i e_{\beta_i}$ and the d sections $\{\sum_{i=1}^n \langle a, \beta_i\rangle z_i e_{\beta_i};\ a \in \mathbf{a}\}$, where $\mathbf{a}$ is an integral unimodular basis of $\mathfrak{t}^*$, generate an appropriate ideal if the parameters $z_1,\dots,z_n$ are sufficiently generic. For Q we take the element $P(z_1e_{\beta_1},\dots,z_ne_{\beta_n})$ of degree d of the homogeneous ring. Then the TRMC may be more precisely formulated as follows: the rational function of the parameters z_i obtained as the toric residue $\mathrm{TorRes}_{\mathbf{f}(z)}P(z_1e_{\beta_1},\dots,z_ne_{\beta_n})$

may be expanded in a domain of $(\mathbb{C}^*)^n$ depending on the chamber $\mathfrak{c}$, and the resulting Laurent series is given by (0.1).

In [15] we gave a proof of this conjecture. The central result to which we reduce the proof of TRMC is a statement about the topology of toric varieties: Let $U(\mathfrak{A})$ be the complement of the complexified hyperplane arrangement in $\mathfrak{a} \otimes_{\mathbb{R}} \mathbb{C}$ induced by the sequence $\mathfrak{A}$. Note that the map $\chi : \mathfrak{a}^* \to H^2(V_{\mathfrak{A}}(\mathfrak{c}), \mathbb{R})$, defined above, may be extended multiplicatively to a map from the space of polynomials on $\mathfrak{a}$ to the even cohomology ring of $V_{\mathfrak{A}}(\mathfrak{c})$. We show that a certain compact real algebraic cycle $Z(\xi) \subset U(\mathfrak{A})$ of dimension r, depending on a parameter $\xi \in \mathfrak{c}$, computes the intersection ring of the toric variety $V_{\mathfrak{A}}(\mathfrak{c})$ in the following sense. For a polynomial Q of degree d on $\mathfrak{a}$, we have

$$\int_{V_{\mathfrak{A}}(\mathfrak{c})} \chi(Q) = \int_{Z(\xi)} \frac{Q\, d\mathrm{vol}}{\prod_{i=1}^n \alpha_i},$$

where $d$vol is the complexification of the volume form on $\mathfrak{a}$ induced by $\mathfrak{a}_{\mathbb{Z}}$. The cycle $Z(\xi)$ is given by the following real algebraic equations

$$Z(\xi) = \left\{ u \in U(\mathfrak{A});\ \prod_{i=1}^n |\alpha_i(u)|^{\langle \alpha_i, \lambda \rangle} = e^{\langle \xi, \lambda \rangle} \text{ for all } \lambda \in \mathfrak{a}_{\mathbb{Z}} \right\}.$$

The proof of this result given in [15] uses a degeneration technique related to the so-called *tropical geometry* [17, 16]. We will demonstrate the argument in a simple example (cf. Example 1).

The Mixed TRMC is a similar naive rational curve counting formula for higher codimensional complete intersections in toric varieties. Here we start with a partition $\{D_1, \dots, D_l\}$ of the index set $\{1, \dots, n\}$ of the sequence $\mathfrak{A}$, which satisfies the following condition: there exists a chamber $\mathfrak{c}$ of $\mathfrak{A}$ such that the vectors $\theta_s = \sum_{i \in D_s} \alpha_i$, $s = 1, \dots, l$ are in the closure of $\mathfrak{c}$. Again, a common smooth generic intersection of zero-sections of the line bundles L_{θ_s}, $s = 1, \dots, l$, is a Calabi-Yau submanifold in $V_{\mathfrak{A}}(\mathfrak{c})$.

Batyrev and Materov [3] formulate their conjecture, the MTRMC, for so-called *nef-partitions*; this notion is a reformulation of the condition we described above.

The structure of the conjectured formula in the mixed case is similar to that of the TRMC. We form a series of the form of (0.1) with a suitably adjusted class K_λ (see (2.15) and (3.5)). Again we need to show that this series converges to a rational function which may be expressed as a toric residue. In this case, the toric residue corresponds the the so-called Cayley polytope instead of the polytope $\Pi^{\mathfrak{B}}$ (cf.[3]). This is a polytope of dimension greater than d; it is constructed from $\Pi^{\mathfrak{B}}$ and the data of a "nef"-partition.

The definitions of the series involved in the Batyrev-Materov conjectures are given by expansions of rational functions, and are relatively simple. It is unclear how to extend our approach to the computation of the Givental J-series [9]. The conjectures of Batyrev-Materov do, however, provide another contribution to the mirror symmetry phenomena.

In this paper, we present a proof of this conjecture. We show, that the MTRMC can again be reduced to our integration formula for toric varieties. Besides this reduction, the paper contains several improvements over [15]. These include

- a more invariant definition of the relevant maps in our construction, which does not rely on the choice of a special $\mathbb{Z}$-basis of $\mathfrak{a}_{\mathbb{Z}}$ (cf. (2.3))
- a more natural and less restrictive condition of regularity on the element ξ, which guarantees our statements on $Z(\xi)$; we call this condition *flag-regularity* (cf. Definitions 2.3, 2.7).
- a more general and simpler Laurent expansion formula which is valid for any chosen basis of $\mathfrak{a}_{\mathbb{Z}}$ (cf. Proposition 2.8).

Now we review the contents of the paper. In §1 we recall the definition of the toric residue and describe a localized formula for it. We apply the idea from [15], which allows us to write the localized formula as a sum of the values of a rational function on $\mathfrak{a}$ over the finite set of solutions of a system of binomial equations. Each of the equations has the form $\prod_{i=1}^{n} \alpha_i^{\nu_i} = t$, where $\nu_i \in \mathbb{Z}$ for $i = 1, \dots, n$, and t is a complex parameter. In §2 we review the main results of [15] with the improvements described above. Finally, in §3, we turn to the proof of the MTRMC of Batyrev and Materov. We start by translating the Cayley construction into the Gale dual language of the sequence $\mathfrak{A}$. This means the following: we add one dimension to the space $\mathfrak{a}$. We denote the extra coordinate on $\mathfrak{a}$ by t. Next we replace $\mathfrak{A}$ with the new collection formed by the vectors $[\alpha_i, 0], i = 1, \dots, n$ together with l new vectors $[-\theta_k, t], k = 1, \dots, l$. Applying our results from the previous section to this augmented sequence, we quickly arrive at the proof of the conjecture.

We end the paper with an appendix, where we justify our use of the local formula, Proposition 1.2, for the toric residue in the mixed case. Here this is rather important, since our toric residue is taken in the Cayley toric variety corresponding to the Cayley polytope, but the deformation parameters vary in a smaller dimensional space. As a result, we cannot claim to be taking generic values of the parameters. This makes checking the conditions of the Propositions more difficult. It is necessary to justify why on the particular subset the assumptions hold. In fact, we compute the exact domain of the parameter space where the conditions hold.

We would like to note that, just as our previous result in [15], we do not need reflexivity. Our formula holds in much greater generality.

We were informed that a few months earlier Kalle Karu [11] has given a proof of this conjecture using different methods.

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1. Local formula for the toric residue

1.1. **Notation and Data.** Let V be a finite-dimensional real vector space of dimension $d+1$ and denote by $V_{\mathbb{C}}$ its complexification. Assume that V is endowed with an integral structure, i.e. with a lattice $V_{\mathbb{Z}}$ of full rank. Then denote by $\mathrm{T}_{\mathbb{C}}(V)$ the complexified torus $V_{\mathbb{C}}/V_{\mathbb{Z}}$, and by $V_{\mathbb{Z}}^*$ the set of linear forms on V which assume integral values on $V_{\mathbb{Z}}$. Thus

$$V_{\mathbb{Z}}^* = \{a \in V^*;\ \langle a, \mu\rangle \in \mathbb{Z} \text{ for all } \mu \in V_{\mathbb{Z}}\}.$$

For each vector $\mu \in V_{\mathbb{Z}}$, denote by e_μ the function $e^{2\pi i\mu}$ on $V_{\mathbb{C}}^*$, which may also be considered as a function on $\mathrm{T}_{\mathbb{C}}(V^*)$.

Further, fix a primitive element $g \in V_{\mathbb{Z}}^*$ and a sequence $\mathfrak{M} = [\mu_1, \mu_2, \ldots, \mu_n]$ of vectors of the lattice $V_{\mathbb{Z}}$, which generate $V_{\mathbb{Z}}$ over $\mathbb{Z}$, and satisfy $\langle g, \mu_i\rangle = 1$, $i = 1, \ldots, n$. Thus $\mathfrak{M}$ is a sequence of vectors in an affine subspace of V of codimension 1. Let $C = C(\mathfrak{M}) \subset V$ be the acute $d+1$-dimensional polyhedral cone generated by $\mathfrak{M}$: $C = \sum_{i=1}^n \mathbb{R}^+\mu_i$. The convex hull $\Pi(\mathfrak{M})$ of the elements of $\mathfrak{M}$ is a convex polytope of dimension d, which serves as the base of the cone C. The faces of the polyhedral cone C are polyhedral sub-cones of C which have dimensions running from 0 to $d+1$. Thus, under our convention, the cone C itself is a face.

Introduce the algebra $S(C)$ with basis given by the functions e_μ, $\mu \in V_{\mathbb{Z}} \cap C$. This algebra is graded:

$$S(C) = \oplus_{k=1}^{\infty} S^k(C) \text{ with } S^k(C) = \bigoplus \mathbb{C}e_\mu,\ \mu \in V_{\mathbb{Z}} \cap C,\ \langle g, \mu\rangle = k.$$

This algebra has a natural ideal $I(C) = \oplus_{k=1}^{\infty} I^k(C)$, the so-called *dualizing ideal*, which is generated by the elements e_μ with μ in the interior of the cone C.

The finitely generated algebra $S(C)$ defines an affine variety $\mathrm{Aff}(C) = \mathrm{Spec}(S(C))$ of dimension $d+1$. A point in $\mathrm{Aff}(C)$ is a character x of $S(C)$, i.e. a ring homomorphism $x : S(C) \to \mathbb{C}$; we will write $e_\mu(x)$ instead of $x(e_\mu)$ and we will use the notation x_i for $e_{\mu_i}(x)$, $i = 1, \ldots, n$. The variety $\mathrm{Aff}(C)$ is a closed cone, and it contains $\mathrm{T}_{\mathbb{C}}(V^*)$ as a Zariski open subset. Indeed, there is a one-to-one correspondence between points of $\mathrm{T}_{\mathbb{C}}(V^*)$ and characters x of $S(C)$ such that $x_i \neq 0$ for all $i = 1, 2, \ldots, n$.

The graded algebra $S(C)$ defines a projective toric variety $\mathrm{Tvar}_g(C)$ of complex dimension d, together with an ample line bundle $L \to \mathrm{Tvar}_g(C)$. Every element $f \in S^k(C)$ defines a section of the line bundle L^k; we will denote this section by the same symbol f. Each such section defines an hypersurface in $\mathrm{Tvar}_g(C)$ via its set of zeros. Denote by H the subspace of V orthogonal to our grading vector g, with lattice $H_{\mathbb{Z}} = H \cap V_{\mathbb{Z}}$. Then H^* is identified with $V^*/\mathbb{R}g$ and the complexified torus $\mathrm{T}_{\mathbb{C}}(H^*) = \mathrm{T}_{\mathbb{C}}(V^*)/\mathbb{C}^*$ is naturally embedded in $\mathrm{Tvar}_g(C)$ as a Zariski open subset.

We choose a primitive element $\gamma \in V_{\mathbb{Z}}$ in the interior of C, and denote by l_γ its degree: $l_\gamma = \langle g, \gamma\rangle$. Denote by $\mathfrak{t}$ the quotient $V/\mathbb{R}\gamma$ endowed with the lattice which is the image of $V_{\mathbb{Z}}$ in this quotient. Then $\mathfrak{t}^*$ is naturally embedded in V^* as the subspace orthogonal to γ. We also have $\mathfrak{t}_{\mathbb{Z}}^* = V_{\mathbb{Z}}^* \cap \mathfrak{t}^*$.

Introduce the auxiliary vector space $\mathfrak{g} = \oplus_{i=1}^{n}\mathbb{R}\omega_i$ endowed with the lattice $\mathfrak{g}_{\mathbb{Z}} = \oplus_{i=1}^{n}\mathbb{Z}\omega_i$. Let $\pi : \mathfrak{g} \to \mathfrak{t} = V/\mathbb{R}\gamma$ be the surjective map which sends ω_i to the image of μ_i modulo $\mathbb{R}\gamma$. We introduce the notation β_i for these image vectors $\pi(\mu_i)$; thus we have a sequence of vectors

$$\mathfrak{B} = [\beta_1, \dots, \beta_n] \subset \mathfrak{t}.$$

The elements of this sequence generate $\mathfrak{t}_{\mathbb{Z}}$ over $\mathbb{Z}$. Note that the natural inclusion $\mathfrak{t}^* \hookrightarrow V^*$ induces the inclusion $\mathrm{T}_{\mathbb{C}}(\mathfrak{t}^*) \hookrightarrow \mathrm{T}_{\mathbb{C}}(V^*)$. We make the obvious observation that

$$\text{if } w \in \mathfrak{t}^*, \text{ then } \langle \beta_i, w\rangle = \langle \mu_i, w\rangle \text{ and } e_{\beta_i}(w) = e_{\mu_i}(w),\ i = 1, \dots, n. \tag{1.1}$$

Denote by $\mathfrak{a}$ the kernel of the map π, endowed as usual by the integral structure inherited from $\mathfrak{g}$; then we have the sequence

$$0 \to \mathfrak{a} \xrightarrow{\iota} \mathfrak{g} \xrightarrow{\pi} \mathfrak{t} \to 0, \tag{1.2}$$

which is exact over $\mathbb{Z}$.

Considering the map $\mathfrak{g} \to V$, which, similarly to π, associates μ_i to ω_i, and denoting its kernel by W, we can arrange our real vector spaces in the following diagram:

$$\begin{tikzcd}
& & \mathfrak{a} \arrow[d, "\iota"'] & H \arrow[d] & \mathbb{R}\gamma \arrow[dl] \\
0 \arrow[r] & W \arrow[r] \arrow[ur] & \mathfrak{g} \arrow[r] \arrow[d, "\pi"'] & V \arrow[r] \arrow[d, "g"] \arrow[dl] & 0 \\
& & \mathfrak{t} & \mathbb{R} &
\end{tikzcd} \tag{1.3}$$

Each of the vector spaces is endowed with an integral structure and the exact sequences which appear on the diagram are exact over $\mathbb{Z}$ as well. The vector sequences we have introduced so far are $\mu_i \in V$, $\beta_i \in \mathfrak{t}$, $\omega_i \in \mathfrak{g}$, $i = 1, \dots, n$.

Note that if $\gamma = \sum_{i=1}^{n} r_i\mu_i$, we can write

$$\mathfrak{a} = W \oplus \mathbb{R}\sum_{i=1}^{n} r_i\omega_i.$$

Lemma 1.1. *Consider the map $H \to \mathfrak{t}$ which is obtained by composing two arrows on the diagram* (1.3)*. This map induces covering maps*

$$\mathrm{T}_{\mathbb{C}}(H) \to \mathrm{T}_{\mathbb{C}}(\mathfrak{t}) \text{ and } \mathrm{T}_{\mathbb{C}}(\mathfrak{t}^*) \to \mathrm{T}_{\mathbb{C}}(H^*)$$

of order l_γ.

Proof. This follows from the dual fact that the image of the vector γ under the linear functional g is l_γ.

□

1.2. **The toric residue.** Let $z = (z_1, z_2, \dots, z_n)$ be a generic vector in $\mathbb{C}^n$, and introduce the element

$$f_z = \sum_{i=1}^{n} z_i e_{\mu_i} \in S^1(C).$$

Considered as a section of the polarizing line bundle L over $\mathrm{Tvar}_g(C)$, it defines a hypersurface $\{f_z = 0\} \subset \mathrm{Tvar}_g(C)$ varying with z. For each $a \in V^*$, we consider the derivative

$$f_{z,a} = \sum_{i=1}^{n} z_i \langle a, \mu_i \rangle e_{\mu_i}$$

of the function f_z in the direction of the vector $a \in V^*$. Again the functions $f_{z,a}$ are elements of $S^1(C)$. Observe that $f_{z,g} = f_z$.

Next, we choose a $\mathbb{Z}$-*basis* $[a_1, \dots, a_d]$ of $\mathfrak{t}^*_{\mathbb{Z}}$; these induce $d+1$ sections $f_{z,0}, f_{z,1}, \dots, f_{z,d}$ of L: $f_{z,0} = f_z$ and $f_{z,j} = f_{z,a_j}$, $j = 1, \dots, d$.

Now consider the following conditions:

$$\{x \in \mathrm{Tvar}_g(C);\ f_{z,j}(x) = 0,\ j = 0, \dots, d\} = \emptyset; \tag{1.4}$$

$$\{x \in \mathrm{Tvar}_g(C);\ f_{z,j}(x) = 0,\ j = 1, \dots, d\} \setminus \mathrm{T}_{\mathbb{C}}(H^*) = \emptyset. \tag{1.5}$$

The first says the hyper-surfaces $\{f_{z,j} = 0\}$, $j = 0, 1, \dots, d$, are in generic position, and the second that the last d intersect inside the complexified torus of $\mathrm{Tvar}_g(C)$. Note that the first condition does not depend on our basis, while the second depends only on the choice of the vector γ. Indeed, we could write the two conditions as

$$\{x \in \mathrm{Tvar}_g(C);\ f_{z,a}(x) = 0 \text{ for } a \in V^*\} = \emptyset;$$
$$\{x \in \mathrm{Tvar}_g(C);\ f_{z,a}(x) = 0 \text{ for } \langle a, \gamma \rangle = 0\} \setminus \mathrm{T}_{\mathbb{C}}(H^*) = \emptyset.$$

Denote by $\mathcal{I}(z)$ the $S(C)$-ideal generated by $\mathbf{f}(z) = \{f_{z,j},\ j = 0, \dots, d\}$. According to [5], if condition (1.4) is fulfilled, then the codimension of $\mathcal{I}(z) \cap I^{d+1}(C)$ in $I^{d+1}(C)$ is exactly 1, and there is a canonical functional defined by Cox [7]:

$$\mathrm{TorRes}_{\mathbf{f}(z)} : I^{d+1}(C) \longrightarrow \mathbb{C}$$

called the *toric residue*, which vanishes on $\mathcal{I}(z) \cap I^{d+1}(C)$.

Now, following [3, Proposition 2.6], we describe a local formula for this toric residue. For $z \in (\mathbb{C}^*)^n$ introduce the set

$$\mathrm{Crit}(\mathfrak{t}^*, z) = \{x \in \mathrm{T}_{\mathbb{C}}(H^*);\ f_{z,j}(x) = 0,\ j = 1, \dots, d\}.$$

Note that if (1.5) holds, then this set coincides with

$$\{x \in \mathrm{Tvar}_g(C);\ f_{z,j}(x) = 0,\ j = 1, \dots, d\},$$

and that if (1.4) holds then $\mathrm{Crit}(\mathfrak{t}^*, z)$ is finite. We will need a variant of $\mathrm{Crit}(\mathfrak{t}^*, z)$:

$$\widetilde{\mathrm{Crit}}(\mathfrak{t}^*, z) = \{w \in \mathrm{T}_{\mathbb{C}}(\mathfrak{t}^*);\ f_{z,j}(w) = 0,\ j = 1, \dots, d\},$$

which is the inverse image of $\mathrm{Crit}(\mathfrak{t}^*, z) \subset \mathrm{T}_{\mathbb{C}}(H^*)$ in the covering group $\mathrm{T}_{\mathbb{C}}(\mathfrak{t}^*)$. According to Lemma 1.1, this set is an l_γ-fold covering of $\mathrm{Crit}(\mathfrak{t}^*, z)$.

Now consider the second derivatives of the function f_z:

$$f_{z,jk} = \sum_{i=1}^{n} z_i \langle a_k, \mu_i \rangle \langle a_j, \mu_i \rangle e_{\mu_i},\ j,k = 1,\dots,d \tag{1.6}$$

and let

$$\mathrm{Hess}_{z,\mathfrak{t}^*} = \det(f_{z,jk})_{j,k\geq 1} \tag{1.7}$$

be the determinant of the matrix formed by these second derivatives. Then $\mathrm{Hess}_{z,\mathfrak{t}^*}$ is an element of $S^d(C)$, which depends on $\mathfrak{t}^*$ and z only, and the product $f_z \,\mathrm{Hess}_{z,\mathfrak{t}^*}$ is an element of $S^{d+1}(C)$.

Consider the following non-degeneracy condition:

$$\mathrm{Hess}_{z,\mathfrak{t}^*}(x) \neq 0 \text{ for each } x \in \mathrm{Crit}(\mathfrak{t}^*, z). \tag{1.8}$$

Now we can formulate the localization theorem for toric residue:

Proposition 1.2. *Let $Q \in I^{d+1}(C)$, and assume that $\mathfrak{t}^*$ and $z \in \mathbb{C}^n$ are such that conditions (1.4), (1.5) and (1.8) hold. Then $Q/(f_z \,\mathrm{Hess}_{z,\mathfrak{t}^*})$ is a meromorphic function on $\mathrm{Tvar}_g(C)$, whose poles avoid $\mathrm{Crit}(\mathfrak{t}^*, z)$, and we have*

$$\mathrm{TorRes}_{\mathbf{f}(z)} Q = l_\gamma \sum_{w \in \widetilde{\mathrm{Crit}}(\mathfrak{t}^*, z)} \frac{Q(w)}{f_z(w)\,\mathrm{Hess}_{z,\mathfrak{t}^*}(w)}. \tag{1.9}$$

This statement is almost identical to [3, Proposition 2.6], and its proof quickly follows, for example, from [6, Theorem 3.2]. The factor l_γ appears for the following reason: on the one hand, the basis $[g, a_1, \dots, a_d]$ is not a $\mathbb{Z}$-basis of $V^*_{\mathbb{Z}}$, but a basis which spans a parallelepiped with volume l_γ in $V^*_{\mathbb{Z}}$; this changes the Hessian by a factor of $(l_\gamma)^2$. On the other hand, we replaced the set $\mathrm{Crit}(\mathfrak{t}^*, z)$ by its l_γ-fold cover in the formula, and this gives us a factor of l_γ^{-1}.

In Propositions 4.5 and 4.6 of the appendix, we will describe a set Z of elements $z \in \mathbb{C}^n$ such that conditions (1.4), (1.5) and (1.8) hold for $z \in Z$.

1.3. **Gale duality.** Now we turn to the description of the set $\mathrm{Crit}(\mathfrak{t}^*, z)$, which is a key point of our work. The situation is similar to that in our article [15].

Consider the exact sequence (1.2). Denote by $\alpha_i \in \mathfrak{a}^*$ the restriction of the coordinate function ω^i on $\mathfrak{g}$ to $\mathfrak{a}$. Let $\mathfrak{A}$ be the sequence

$$\mathfrak{A} := [\alpha_1, \alpha_2, \dots, \alpha_n].$$

Then the sequence $\mathfrak{A}$ is defined to be *Gale dual* of the sequence $\mathfrak{B}$.

Observe that the convex hull of the vectors in $\mathfrak{B}$ contains the origin in its interior, and as a result, the vectors of the Gale dual sequence $\mathfrak{A}$ lie in an open half-space of $\mathfrak{a}^*$. We called such a sequence $\mathfrak{A}$ *projective* in [15].

An element u of $\mathfrak{a}$ is written $u = \sum_{i=1}^n \alpha_i(u)\omega_i$ and for every $u \in \mathfrak{a}$, we have

$$\sum_{i=1}^n \alpha_i(u)\mu_i \in \mathbb{R}\gamma.$$

The sequence $\mathfrak{A}$ defines a hyperplane arrangement in $\mathfrak{a}_\mathbb{C}$ and we denote by $U(\mathfrak{A})$ the complement of this hyperplane arrangement:

$$U(\mathfrak{A}) = \{u \in \mathfrak{a}_\mathbb{C};\ \alpha(u) \neq 0 \text{ for all } \alpha \in \mathfrak{A}\}.$$

For $\lambda \in \mathfrak{a}_\mathbb{Z}$, we introduce the rational functions

$$p_\lambda(u) = \prod_{i=1}^n \alpha_i(u)^{\langle \alpha_i, \lambda\rangle} \text{ and } z^\lambda = \prod_{i=1}^n z_i^{\langle \alpha_i, \lambda\rangle}. \tag{1.10}$$

on $\mathfrak{a}_\mathbb{C}$ and $(\mathbb{C}^*)^n$, respectively.

Now, similarly to [15], we parameterize the set $\widetilde{\mathrm{Crit}}(\mathfrak{t}^*, z) \subset \mathrm{T}_\mathbb{C}(\mathfrak{t}^*)$ by a finite subset $O(z, \mathfrak{A})$ in $\mathfrak{a}_\mathbb{C}$.

Definition 1.1. For $z \in (\mathbb{C}^*)^n$ define the set

$$O(z, \mathfrak{A}) := \{u \in U(\mathfrak{A});\ p_\lambda(u) = z^\lambda \text{ for all } \lambda \in \mathfrak{a}_\mathbb{Z}\}.$$

Lemma 1.3. *If $u \in O(z, \mathfrak{A})$, then there exists a unique element $w \in \widetilde{\mathrm{Crit}}(\mathfrak{t}^*, z)$ such that*

$$z_i e_{\beta_i}(w) = \alpha_i(u) \quad \textit{for } i = 1, \ldots, n. \tag{1.11}$$

This correspondence between $O(z, \mathfrak{A})$ to $\widetilde{\mathrm{Crit}}(\mathfrak{t}^*, z)$ is bijective.

Proof. If w is in $\widetilde{\mathrm{Crit}}(\mathfrak{t}^*, z)$, then $f_{z,a}(w) = 0$ for all $a \in \mathfrak{t}^*$. Explicitly,

$$\sum_{i=1}^n z_i \langle a, \beta_i\rangle e_{\beta_i}(w) = 0 \text{ for all } a \in \mathfrak{t}^*.$$

Thus

$$\sum_{i=1}^n z_i e_{\beta_i}(w)\beta_i = 0.$$

By definition of Gale duality, this condition means that there exists $u \in \mathfrak{a}_\mathbb{C}$ such that $z_i e_{\beta_i}(w) = \alpha_i(u)$. We thus have associated to $w \in \widetilde{\mathrm{Crit}}(\mathfrak{t}^*, z)$ an element $u \in \mathfrak{a}$. If none of the coordinates z_i of z vanishes, then $\alpha_i(u) \neq 0$ for $i = 1, \ldots, n$, and, consequently, $u \in U(\mathfrak{A})$.

Now consider the point u associated to w this way, and let $\lambda = \sum_{i=1}^n n_i\omega_i$ be an element in $\mathfrak{a}_\mathbb{Z}$, i.e. $\sum_{i=1}^n n_i\mu_i \in \mathbb{R}\gamma$. Then

$$\prod_{i=1}^n \alpha_i(u)^{n_i} = \prod_{i=1}^n z_i^{n_i} e_{\sum_{i=1}^n n_i\beta_i}(w) = z^\lambda,$$

since $w \in \mathrm{T}_\mathbb{C}(\mathfrak{t}^*)$. Thus we obtain that the equation $p_\lambda(u) = z^\lambda$ holds for all $\lambda \in \mathfrak{a}_\mathbb{Z}$, i.e. $u \in O(z, \mathfrak{A})$.

Conversely, if $u \in O(z, \mathfrak{A})$, then, as shown in ([15], Lemma 4.3), we can find $w \in \mathrm{T}_{\mathbb{C}}(\mathfrak{t}^*)$ such that $z_i e_{\beta_i}(w) = \alpha_i(u)$; this element w is in $\widetilde{\mathrm{Crit}}(\mathfrak{t}^*, z)$. □

Now we turn to (1.9); our next step is to express the values $f_z(w)$ and $\mathrm{Hess}_{z,\mathfrak{t}^*}(w)$ for $w \in \widetilde{\mathrm{Crit}}(\mathfrak{t}^*, z)$ as values of certain functions on $\mathfrak{a}_{\mathbb{C}}$ evaluated at the corresponding point u. Introduce the polynomial G on $\mathfrak{a}_{\mathbb{C}}$:

$$G(u) = \sum_{\nu} \mathrm{vol}_{\mathfrak{B}}(\nu)^2 \prod_{j \notin \nu} \alpha_j(u), \tag{1.12}$$

where ν runs over the subsets of cardinality d of the set $\{1, 2, \dots, n\}$. In this equation the volume $\mathrm{vol}_{\mathfrak{B}}(\nu)$ is the volume of the parallelepiped $\sum_{j\in\nu}[0,1]\beta_j$ in $\mathfrak{t}$. In particular, in the sum in (1.12) only those subsets ν will contribute for which the vectors $\{\beta_j, j \in \nu\}$ are linearly independent. Incidentally, these are exactly the subsets for which the vectors $\{\alpha_j,\ j \notin \nu\}$ are linearly independent (cf. [15, Proposition 1.2]).

We will denote by $w(u)$ the point of $\widetilde{\mathrm{Crit}}(\mathfrak{t}^*, z)$ associated to a point $u \in O(z, \mathfrak{A})$, i.e. when w and u are related by (1.11).

Lemma 1.4. *If $w = w(u)$, then*

$$f_z(w(u)) = \sum_{i=1}^{n} \alpha_i(u),$$

$$\mathrm{Hess}_{z,\mathfrak{t}^*}(w(u)) = G(u).$$

Proof. The first formula follows immediately from the definition of $f_z = \sum_{i=1}^n z_i e_{\mu_i}$ and equation (1.11). The second formula is proved in [15, Proposition 4.5]. □

We introduce the notation $\kappa_{\mathfrak{A}} = \sum_{i=1}^n \alpha_i \in \mathfrak{a}^*$.

Proposition 1.5. *Let P be a homogeneous polynomial of degree $d - l_\gamma + 1$ in n variables; then $e_\gamma P(z_1 e_{\mu_1}, \dots, z_n e_{\mu_n})$ is an element of $I^{d+1}(C)$. If $\gamma \in V_{\mathbb{Z}}^*$ and $z \in \mathbb{C}^n$ satisfy conditions* (1.4), (1.5) *and* (1.8), *then*

$$\mathrm{TorRes}_{\mathbf{f}(z)}\, e_\gamma P(z_1 e_{\mu_1}, \dots, z_n e_{\mu_n}) = l_\gamma \sum_{u \in O(z,\mathfrak{A})} \frac{P(\alpha_1(u), \dots, \alpha_n(u))}{\kappa_{\mathfrak{A}}(u) G(u)}. \tag{1.13}$$

Proof. Using Proposition 1.2 and (1.1), we may write

$$\mathrm{TorRes}_{\mathbf{f}(z)}\, e_\gamma P(z_1 e_{\mu_1}, \dots, z_n e_{\mu_n}) = l_\gamma \sum_{w \in \widetilde{\mathrm{Crit}}(\mathfrak{t}^*, z)} \frac{e_\gamma(w) P(z_1 e_{\beta_1}(w), \dots, z_n e_{\beta_n}(w))}{f_z(w)\, \mathrm{Hess}_{z,\mathfrak{t}^*}(w)}.$$

Note that the character e_γ is identically 1 on $\mathrm{T}_{\mathbb{C}}(\mathfrak{t}^*)$, thus Lemma 1.4 implies the Proposition. □

2. Summary of our earlier results

2.1. **Flags and iterated residues.** Our goal in this section is to develop a Laurent-type expansion for the functions of the complex variable $z = (z_1, \ldots, z_n)$, which appear on the right hand side of (1.13). These are essentially results of [15], but the exposition is improved here. In particular, we present an invariant formalism, which avoids the choice of a special ($\mathfrak{c}$-positive) basis of $\mathfrak{a}_{\mathbb{Z}}$. We also introduce a more precise notion of regularity of our target vector ξ.

We maintain some of the notation of the previous section. Thus we have an exact sequence

$$0 \longrightarrow \mathfrak{a} \xrightarrow{\iota} \mathfrak{g} \xrightarrow{\pi} \mathfrak{t} \longrightarrow 0$$

of real vector spaces of dimensions r, n and d; thus $n - d = r$. The space $\mathfrak{g}$ has a fixed basis $(\omega_1, \ldots, \omega_n)$, and we have three lattices

$$\mathfrak{g}_{\mathbb{Z}} = \oplus_{i=1}^{n} \mathbb{Z}\omega_i,\ \mathfrak{t}_{\mathbb{Z}} = \pi(\mathfrak{g}_{\mathbb{Z}})\ , \mathfrak{a}_{\mathbb{Z}} = \ker(\pi|_{\mathfrak{g}_{\mathbb{Z}}}).$$

This construction gives rise to the Gale dual sequences of vectors

$$\mathfrak{A} = [\alpha_1, \ldots, \alpha_n] \subset \mathfrak{a}_{\mathbb{Z}}^*, \quad \mathfrak{B} = [\beta_1, \ldots, \beta_n] \subset \mathfrak{t}_{\mathbb{Z}}.$$

Denote by $\mathbb{C}_{\mathfrak{A}}[\mathfrak{a}]$ the graded linear space of rational functions on $\mathfrak{a}_{\mathbb{C}}$ whose denominators are products of powers of elements of $\mathfrak{A}$. The functions in $\mathbb{C}_{\mathfrak{A}}[\mathfrak{a}]$ are regular on $U(\mathfrak{A})$, the complement of the hyperplane arrangement induced by $\mathfrak{A}$. Recall that for each $\lambda \in \mathfrak{a}_{\mathbb{Z}}$, we defined a rational function $p_\lambda \in \mathbb{C}_{\mathfrak{A}}[\mathfrak{a}]$ in (1.10) of homogeneous degree $\langle \kappa_{\mathfrak{A}}, \lambda \rangle$, where $\kappa_{\mathfrak{A}} = \sum_{i=1}^{n} \alpha_i$.

We fix an orientation of $\mathfrak{a}$, and choose an oriented $\mathbb{Z}$-basis $\boldsymbol{\lambda} = [\lambda_1, \ldots, \lambda_r]$ of $\mathfrak{a}_{\mathbb{Z}}$. We denote by $d$vol the holomorphic r-form $d\gamma_1 \wedge d\gamma_2 \wedge \cdots \wedge d\gamma_r$ on $\mathfrak{a}_{\mathbb{C}}$, where $[\gamma_1, \ldots, \gamma_r] \subset \mathfrak{a}^*$ is the basis dual to $\boldsymbol{\lambda}$. This form depends only on the orientation and the integral structure of $\mathfrak{a}$, and not on the choice of the basis $\boldsymbol{\lambda}$.

Introduce the notation $p_j = p_{\lambda_j}$ and $q_j = z^{\lambda_j}$, and define the maps

$$p_{\boldsymbol{\lambda}} = (p_1, p_2, \ldots, p_r) : U(\mathfrak{A}) \to (\mathbb{C}^*)^r \tag{2.1}$$

and

$$q_{\boldsymbol{\lambda}} = (q_1, q_2, \ldots, q_r) : (\mathbb{C}^*)^n \to (\mathbb{C}^*)^r.$$

The map $p_{\boldsymbol{\lambda}}$ plays a central role in our investigations. We will need a formula ([15, Proposition 4.5]) for the Jacobian of this map:

Lemma 2.1. *We have the equality of meromorphic r forms on $U(\mathfrak{A})$:*

$$\frac{dp_1}{p_1} \wedge \frac{dp_2}{p_2} \wedge \cdots \wedge \frac{dp_r}{p_r} = \frac{G(u)\, d\mathrm{vol}}{\prod_{i=1}^{n} \alpha_i(u)}, \tag{2.2}$$

where the function $G(u)$ is defined in (1.12).

There is, in fact, a more invariant way to describe the map $p_{\boldsymbol{\lambda}}$. Let $p : U(\mathfrak{A}) \to \mathrm{T}_{\mathbb{C}}(\mathfrak{a}^*)$ be the equidimensional map given by the formula

$$p(u) = \frac{1}{2\pi\sqrt{-1}} \sum_{i=1}^{n} \log(\alpha_i(u))\, \alpha_i.$$

The map formally depends on the branch of the logarithm one chooses, however, this choice is immaterial in the quotient $\mathrm{T}_{\mathbb{C}}(\mathfrak{a}^*) = \mathfrak{a}^*_{\mathbb{C}}/\mathfrak{a}^*_{\mathbb{Z}}$.

Similarly let $q : (\mathbb{C}^*)^n \to \mathrm{T}_{\mathbb{C}}(\mathfrak{a}^*)$ be the map given by the formula

$$q(z) = \frac{1}{2\pi\sqrt{-1}} \sum_{i=1}^{n} \log(z_i)\, \alpha_i.$$

If $\lambda \in \mathfrak{a}_{\mathbb{Z}}$, then $e_\lambda(p(u)) = p_\lambda(u)$ and $e_\lambda(q(z)) = z^\lambda$. Thus the set $O(z, \mathfrak{A})$ (see Definition 1.1) may be given as the set of solutions of the equation on $U(\mathfrak{A})$:

$$O(z, \mathfrak{A}) = \{u \in U(\mathfrak{A});\, p(u) = q(z)\};$$

one can also write $O(z, \mathfrak{A}) = p^{-1}(q(z))$.

Using the $\mathbb{Z}$-basis $\boldsymbol{\lambda} = [\lambda_1, \dots, \lambda_r]$ of $\mathfrak{a}_{\mathbb{Z}}$, we identify $\mathrm{T}_{\mathbb{C}}(\mathfrak{a}^*)$ with $(\mathbb{C}^*)^r$ via the map $(e_{\lambda_1}, \dots, e_{\lambda_r})$. After this identification, the map $p_{\boldsymbol{\lambda}}$ is the map p and $q_{\boldsymbol{\lambda}}$ is the map q. We collected the maps we will use in the following diagram:

$$\begin{tikzcd}
 & U(\mathfrak{A}) \arrow[dl, "p_{\boldsymbol{\lambda}}"'] \arrow[dd, "\iota"] \arrow[dr, "\Psi"] \arrow[drr, "p"] & & \\
(\mathbb{C}^*)^r & & \mathfrak{a}^* & \mathrm{T}_{\mathbb{C}}(\mathfrak{a}^*) \arrow[l, "\mathrm{Im}'"] \\
 & (\mathbb{C}^*)^n \arrow[ul, "q_{\boldsymbol{\lambda}}"] \arrow[ur, "\psi"] \arrow[urr, "q"'] & &
\end{tikzcd} \tag{2.3}$$

- ι The map from $U(\mathfrak{A})$ to $(\mathbb{C}^*)^n$ is given by $u \mapsto (\alpha_1(u), \dots, \alpha_n(u))$; it is the restriction of the map ι in (1.2);
- Im' is the map $w \mapsto -\mathrm{Re}(2\pi\sqrt{-1}w)$ from $\mathfrak{a}^*_{\mathbb{C}}/\mathfrak{a}^*_{\mathbb{Z}}$ to $\mathfrak{a}^*$;
- $\psi : (z_1, \dots, z_n) \mapsto -\sum_{i=1}^{n} \log|z_i|\, \alpha_i$.
- $\Psi(u) = \psi(\iota(u)) = \mathrm{Im}'(p(u)) = -\sum_{i=1}^{n} \log(|\alpha_i(u)|)\, \alpha_i(u)$.

The following statement is straightforward:

Lemma 2.2. *The maps ι and Im' are proper.*

To formulate our expansion formula, we need some preparations. We introduce the following notions from [15]. Let $\mathcal{FL}(\mathfrak{A})$ be the finite set of flags

$$F = [F_0 = \{0\} \subset F_1 \subset F_2 \subset \cdots \subset F_{r-1} \subset F_r = \mathfrak{a}^*], \text{ where } \dim F_j = j,$$

such that the finite sequence $\mathfrak{A} \cap F_j$ spans F_j for each $j = 1, \dots, r$. For each $F \in \mathcal{FL}(\mathfrak{A})$, we choose an ordered basis $(\gamma_1^F, \gamma_2^F, \dots, \gamma_r^F)$ of $\mathfrak{a}^*$, which is unimodular with respect to the volume form $d\mathrm{vol}$ on $\mathfrak{a}$, and which is

such that $F_j = \oplus_{k=1}^{j} \mathbb{R}\gamma_k^F$. If F is fixed, then we will use the simplified notation $u_j = \langle \gamma_j^F, u\rangle$, $j = 1, \dots, r$. We call the functionals u_j, $j = 1, \dots, r$, coordinates *adapted to* F.

Now let $F \in \mathcal{FL}(\mathfrak{A})$, and let N be a positive real number. Define the open subset $U(F, N)$ of $\mathfrak{a}_{\mathbb{C}}$ by

$$U(F, N) = \{u \in \mathfrak{a}_{\mathbb{C}}\,;\ 0 < N|u_j| < |u_{j+1}|,\ j = 1, 2, \dots, r-1\}.$$

We make the following simple observations:

Lemma 2.3. (1) *If N is sufficiently large, then the sets $\{U(F, N);\ F \in \mathcal{FL}(\mathfrak{A})\}$ are disjoint, moreover, for each $F \in \mathcal{FL}(\mathfrak{A})$, $U(F, N) \subset U(\mathfrak{A})$, and the linear form $\kappa_{\mathfrak{A}} = \sum_{i=1}^{n} \alpha_i$ does not vanish on $U(F, N)$.*

(2) *The set $U(F, N)$ is invariant under the scaling $u \to e^t u$.*

(3) *If $N_2 > N_1$, then $U(F, N_2) \subset U(F, N_1)$.*

(4) *For N sufficiently large, $U(F, N)$ is isomorphic to the product of $\mathbb{C}^*$ with $r-1$ small punctured disks. Choose a sequence of real numbers $\boldsymbol{\epsilon} : 0 < \epsilon_1 \ll \epsilon_2 \ll \cdots \ll \epsilon_r$, where $\epsilon \ll \delta$ means $N\epsilon < \delta$. Then the embedded torus*

$$T_F(\boldsymbol{\epsilon}) = \{u \in \mathfrak{a}_{\mathbb{C}};\ |u_j| = \epsilon_j,\ j = 1, \dots, r\} \subset U(F, N) \tag{2.4}$$

oriented by the form $d\arg u_1 \wedge \cdots \wedge d\arg u_r$, represents a generator of $H_r(U(F, N), \mathbb{Z})$.

Definition 2.1. Let $F \in \mathcal{FL}(\mathfrak{A})$, and choose N large enough in order to have $U(F, N) \subset U(\mathfrak{A})$. We denote by $h(F)$ the homology class in $H_r(U(\mathfrak{A}), \mathbb{Z})$ of the oriented compact torus $T_F(\boldsymbol{\epsilon})$.

If ω is a holomorphic r-form on $U(F, N)$, then it is closed. Thus we can write $\int_{h(F)} \omega$ for the integral $\int_{T_F(\boldsymbol{\epsilon})} \omega$, since it only depends on the homology class $h(F)$ of the cycle $T_F(\boldsymbol{\epsilon})$.

If $\omega = \phi(u) d\mathrm{vol}$, where $\phi \in \mathbb{C}_{\mathfrak{A}}[\mathfrak{a}]$, then $\frac{1}{(2\pi\sqrt{-1})^r} \int_{h(F)} \phi(u) d\mathrm{vol}$ coincides with the *iterated residue* of ϕ with the respect to F ([13]). This is defined as follows: write ϕ as a rational function ϕ^F of the coordinates u_j and define the iterated residue as

$$\operatorname*{Res}_F(\phi) = \operatorname*{Res}_{u_r=0} du_r \operatorname*{Res}_{u_{r-1}=0} \cdots \operatorname*{Res}_{u_1=0} du_1\ \phi^F(u_1, u_2, \dots, u_r), \tag{2.5}$$

where each residue is taken assuming that the variables with higher indices have a fixed, nonzero value.

Let $F \in \mathcal{FL}(\mathfrak{A})$ be a flag. Introduce the vectors

$$\kappa_j^F = \sum \{\alpha_i;\ i = 1, \dots, n,\ \alpha_i \in F_j\},\ j = 1, \dots, r.$$

Note that κ_r^F is independent of F and equals $\kappa_{\mathfrak{A}} = \sum_{i=1}^{n} \alpha_i$.

Definition 2.2.
- We say that a flag $F \in \mathcal{FL}(\mathfrak{A})$ is *proper* if the vectors κ_j^F, $j = 1, \dots, r$, are linearly independent.
- Assuming F is proper, define the sign $\nu(F) = \pm 1$ depending on whether the sequence $[\kappa_1^F, \dots, \kappa_r^F]$ is positively oriented or not.

- For a flag $F \in \mathcal{FL}(\mathfrak{A})$, introduce the non-acute cone $\mathfrak{s}(F, \mathfrak{A})$ generated by the non-negative linear combinations of the elements $\{\kappa_j^F, j = 1, \ldots, r-1\}$ and the line $\mathbb{R}\kappa_{\mathfrak{A}}$:
$$\mathfrak{s}(F, \mathfrak{A}) = \sum_{j=1}^{r-1} \mathbb{R}^{\geq 0}\kappa_j^F + \mathbb{R}\kappa_{\mathfrak{A}}$$
- For $\xi \in \mathfrak{a}^*$ denote by $\mathcal{FL}(\mathfrak{A}, \xi)$ the set of flags $F \in \mathcal{FL}(\mathfrak{A})$ such that $\xi \in \mathfrak{s}(F, \mathfrak{A})$.

Remark 2.1. Note that for any $t \in \mathbb{R}$, we have $\mathcal{FL}(\mathfrak{A}, \xi) = \mathcal{FL}(\mathfrak{A}, \xi + t\kappa_{\mathfrak{A}})$.

In [15], we introduced the notion of τ-regularity for a vector $\xi \in \mathfrak{a}^*$. In the present paper we replace it with a less restrictive notion of being removed from the boundary inside the cones $\mathfrak{s}(F, A)$, $F \in \mathcal{FL}(\mathfrak{A}, \xi)$.

Definition 2.3. We define a vector $\xi \in \mathfrak{a}^*$ to be $\mathcal{FL}(\mathfrak{A})$*-regular* if for any flag $F \in \mathcal{FL}(\mathfrak{A}, \xi)$ the first $r-1$ coefficients, $m_1, \ldots, m_{r-1}$, in any linear expression
$$\xi = m_1\kappa_1^F + \cdots + m_{r-1}\kappa_{r-1}^F + m_r\kappa_r^F$$
do not vanish. Given a positive number τ, we will call ξ $\mathcal{FL}(\mathfrak{A})_\tau$*-regular* if in any such linear expression we have $\min_{1 \leq j \leq r-1} m_j > \tau$. We denote by $\mathfrak{a}^*_{[\tau]}$ the subset of $\mathcal{FL}(\mathfrak{A})_\tau$-regular elements of $\mathfrak{a}^*$.

Remark 2.2.
1. Note that in the definition, $F \in \mathcal{FL}(\mathfrak{A}, \xi)$ implies that $m_j \geq 0$, $j = 1, \ldots, r$
2. It is easy to see that if ξ is a $\mathcal{FL}(\mathfrak{A})$-regular element of $\mathfrak{a}^*$, then every flag $F \in \mathcal{FL}(\mathfrak{A}, \xi)$ is proper.
3. If ξ is a $\mathcal{FL}(\mathfrak{A})_\tau$-regular, then for any $t \in \mathbb{R}$, the element $\xi + t\kappa_{\mathfrak{A}}$ is also $\mathcal{FL}(\mathfrak{A})_\tau$-regular.
4. The set $\mathfrak{a}^*_{[\tau]}$ is a disjoint union of open cones in $\mathfrak{a}^*$. It occupies "most" of the vector space $\mathfrak{a}^*$, in the sense that the intersection of its complement with a generic affine line $L \subset \mathfrak{a}^*$ is bounded.

Consider the maps on the diagram (2.3). For $\xi \in \mathfrak{a}^*$, define the set

$$Z(\xi) = \left\{ u \in U(\mathfrak{A});\ \sum_{i=1}^{n} \log|\alpha_i(u)|\alpha_i = -\xi \right\} \tag{2.6}$$

Observe that
$$Z(\xi) = \Psi^{-1}(\xi) = (\mathrm{Im}' \circ p)^{-1}(\xi) = (\psi \circ \iota)^{-1}(\xi).$$

Choosing an oriented $\mathbb{Z}$-basis $\boldsymbol{\lambda} = [\lambda_1, \ldots, \lambda_r]$ of $\mathfrak{a}_{\mathbb{Z}}$, we can present the set $Z(\xi)$ as the set of common solutions of the r analytic equations on $U(\mathfrak{A})$:
$$Z(\xi) = \{|p_j(u)| = e^{-\langle \xi, \lambda_j \rangle};\ j = 1, \ldots, r\};$$
thus $Z(\xi)$ is the inverse image of the product of r circles under the map $p : U(\mathfrak{A}) \to \mathrm{T}_{\mathbb{C}}(\mathfrak{a}^*) = (\mathbb{C}^*)^r$. In particular, $Z(\xi)$ is a real analytic subvariety of $U(\mathfrak{A})$. The form $d\arg p_1 \wedge \cdots \wedge d\arg p_r$ defines an orientation of $Z(\xi)$; this

orientation depends only on the orientation of $\mathfrak{a}$ and not on the particular basis oriented $\boldsymbol{\lambda}$ we picked.

Example 1. Consider the case $d = r = 2, n = 4$, $\mathfrak{a} = \mathbb{R}^2$, $\mathfrak{a}_{\mathbb{Z}} = \mathbb{Z}^2$, with coordinates x, y. Let $\mathfrak{A} = (y, x+y, y, x)$. Then $\kappa = 2x + 3y$, and κ is in the interior of the chamber $\{mx + ny;\ 0 < m < n\}$, which corresponds to the Hirzebruch surface, the blow-up of $\mathbb{P}^2$ at one point. Pick the vector $\xi = (\log \epsilon)(x + 2y)$ in this chamber, where ϵ is a small positive constant. Then the equations defining $Z(\xi)$ read as

$$|x(x+y)| = \epsilon, \quad |y^2(x+y)| = \epsilon^2. \tag{2.7}$$

$\square$

With these preparations, we are ready to formulate the main results of [15]. They were proved assuming the notion of τ-regularity for a vector $\xi \in \mathfrak{a}^*$ in the strong sense of Definition 2.2 of [15]. It is easy to see, however, that the results, as well as the proofs of [15] remain true under the weaker assumption of $\mathcal{FL}(\mathfrak{A})_\tau$-regularity.

Definition 2.4. We say that a continuous map $\tilde{p} : U \to V$ is *proper to* an open subset $V' \subset V$ if for every compact $K \subset V'$ the inverse image $\tilde{p}^{-1}(K)$ is compact.

Theorem 2.4. (1) *For any sufficiently large $N > 0$ and any proper flag $F \in \mathcal{FL}(\mathfrak{A})$, the holomorphic map p, restricted to $U(F, N)$ is non-singular (*[15, §5.3]*).*

(2) *For τ sufficiently large, the map $\Psi : U(\mathfrak{A}) \to \mathfrak{a}^*$ is proper to the set $\mathfrak{a}^*_{[\tau]}$ of $\mathcal{FL}(\mathfrak{A})_\tau$-regular vectors in $\mathfrak{a}^*$.*

(3) *Given $N > 0$, there is $\tau > 0$ such that if $\xi \in \mathfrak{a}^*_{[\tau]}$ then*

$$Z(\xi) \subset \cup_{F \in \mathcal{FL}(\mathfrak{A}, \xi)} U(F, N). \tag{2.8}$$

It will be convenient to make the following

Definition 2.5. Fix N sufficiently large to satisfy the conditions of Lemma 2.3 (1) and Theorem 2.4 (1), and let τ be such that it satisfy the conditions of Theorem 2.4 (2) and also statement (3) of the same Theorem with respect to N. Then we will call the pair of positive constants (N, τ) *sufficient*.

Let us point out some corollaries of the Theorem.

Corollary 2.5. (1) *The map p is proper to the set $\mathrm{Im}'^{-1}(\mathfrak{a}^*_{[\tau]})$. Similarly, the map $p_{\boldsymbol{\lambda}}$ is proper to the set of vectors of the form*

$$\left\{(q_1, \dots, q_r);\ |q_j| = e^{-\langle \xi, \lambda_j \rangle},\ j = 1, \dots, r \ \text{ for some } \xi \in \mathfrak{a}^*_{[\tau]}\right\}.$$

(2) *For sufficient (τ, N) and any $z \in (\mathbb{C}^*)^n$ such that $\psi(z) \in \mathfrak{a}^*_{[\tau]}$, the set $O(z, \mathfrak{A})$ is finite, and has the following decomposition into a disjoint union:*

$$O(z, \mathfrak{A}) = \bigcup \{O(z, \mathfrak{A}) \cap U(F, N);\ F \in \mathcal{FL}(\mathfrak{A}, \psi(z))\}.$$

(3) *For sufficient* (N,τ) *and any* $\xi \in \mathfrak{a}^*_{[\tau]}$*, the cycle* $Z(\xi)$ *is smooth and compact in* $U(\mathfrak{A})$.

The first statement follows from Lemma 2.2 and Theorem 2.4. To prove the second statement, recall that

- $O(z,\mathfrak{A}) = p^{-1}(q(z))$,
- for $\xi \in \mathfrak{a}^*_{[\tau]}$ every flag in $\mathcal{FL}(\mathfrak{A},\xi)$ is proper, and
- $\mathrm{Im}'^{-1}(\mathfrak{a}^*_{[\tau]}) = q\left(\psi^{-1}(\mathfrak{a}^*_{[\tau]})\right)$.

Thus we represented $O(z,\mathfrak{A})$ as the inverse image of a point under a proper non-singular equidimensional map. This implies that $O(z,\mathfrak{A})$ is finite. The second part of statement (2) follows from Theorem 2.4 (3).

Finally, Statement (3) of Corollary 2.5 follows from a similar argument, since $Z(\xi)$ is the inverse image of a smooth torus under p.

Remark 2.3. We would like to emphasize that, in general, the map p is *not* proper on the whole of $U(\mathfrak{A})$, and this is why the homology class of the cycle $Z(\xi)$ may vary with ξ.

Once we know that the cycle $Z(\xi)$ is compact and smooth in $U(\mathfrak{A})$, it is natural to try to compute its homology class in $H_r(U(\mathfrak{A}),\mathbb{Z})$. The method of this computation relies on a degeneration technique, which is known as *tropical geometry* [17, 16]. Let us explain this on our Example 1. We consider (2.7), let the parameter ϵ be sufficiently small, and introduce the tropical ansatz

$$|x| = \epsilon^a,\ |y| = \epsilon^b,\ |x+y| = \epsilon^c.$$

Then the equations (2.7) imply the equalities

$$\begin{cases} a+c &= 1, \\ 2b+c &= 2. \end{cases} \tag{2.9}$$

These, naturally, do not determine a, b and c, but we observe that for small enough ϵ, if $|x+y|$ is very small compared to $|x|$, then $|x|$ and $|y|$ should be rather close. These gives us the following three possibilities:

(1) $c > a, b$, which implies $a \sim b$
(2) $b > a, c$, which implies $a \sim c$
(3) $a > c, b$, which implies $c \sim b$.

Here by $>$, we mean *significantly greater*, and by $\sim$ we mean *very close.* Now we can go back to our system (2.9), and solve them under the three possible conditions $a = b$, $a = c$, or $b = c$, and obtain the three solutions

$$(1,1,0),\quad (1/2,3/4,1/2),\quad (1/3,2/3,2/3).$$

However, only the second of these equations satisfies the corresponding inequalities. Thus we, informally, conclude, that for $\xi = x+2y$ the cycle $Z_\epsilon(\xi)$ consists of a torus which is very close to the torus $\{|y| = \epsilon^{3/4},\ |x| = \epsilon^{1/2}\} \subset$

$\mathbb{C}^2$. Integration over such a torus is equivalent to a single iterated residue:

$$\mathrm{JK}_{\mathfrak{c}(\kappa)}(f) = \operatorname*{Res}_x \operatorname*{Res}_y f\,dx\,dy.$$

The result in the general case is the following

Theorem 2.6. *For a sufficient pair (N,τ) and $\xi \in \mathfrak{a}^*_{[\tau]}$, the homology class $[Z(\xi)\cap U(F,N)] \in H_r(U(\mathfrak{A}),\mathbb{Z})$ is equal to $\nu(F)h(F)$, (cf. Definitions 2.2,2.1). Hence, using Theorem 2.4 (3), we can conclude that*

$$[Z(\xi)] = \sum_{F\in\mathcal{FL}(\mathfrak{A},\xi)} \nu(F)h(F) \in H_r(U(\mathfrak{A}),\mathbb{Z}), \tag{2.10}$$

Recall that Lemma 1.3 establishes a bijection $w \to u$ between $\widetilde{\mathrm{Crit}}(z)$ and $O(z,\mathfrak{A})$, and that under this bijection the value of the function $f_z \operatorname{Hess}_{z,\mathfrak{t}^*}$ at w coincides with the value of the function $\kappa_{\mathfrak{A}}G$ at u (Lemma 1.4). The function $\kappa_{\mathfrak{A}}(u)G(u)$ does not vanish on $U(F,N)$ provided N is sufficiently large and F is a proper flag. The fact that $\kappa_{\mathfrak{A}}$ does not vanish on $U(F,N)$ for large N is easy and stated in Lemma 2.3. The fact that $G(u)$ does not vanish on $U(F,N)$ provided N is sufficiently large and F is proper is the content of Proposition 5.9 of [15]. Thus we obtain the following.

Corollary 2.7. *Let $z \in (\mathbb{C}^*)^n$ such that $\psi(z)$ is $\mathcal{FL}(\mathfrak{A})_\tau$-regular. Then $\widetilde{\mathrm{Crit}}(z)$ is finite and the function $f_z \operatorname{Hess}_{z,\mathfrak{t}^*}$ does not vanish at any of the points of $\widetilde{\mathrm{Crit}}(z) \subset \mathrm{Tvar}_g(C)$.*

Proposition 2.8. *Let (τ,N) be a sufficient pair of constants, and let $z = (z_1,z_2,\dots,z_n) \in (\mathbb{C}^*)^n$ be such that $\psi(z)$ is $\mathcal{FL}(\mathfrak{A})_\tau$-regular. Then for any $F \in \mathcal{FL}(\mathfrak{A},\psi(z))$ and any holomorphic function ϕ on $U(F,N)$, we have*

$$\sum_{u\in O(z,\mathfrak{A})\cap U(F,N)} \phi(u) = \frac{\nu(F)}{(2\pi\sqrt{-1})^r} \sum_{\lambda\in\mathfrak{a}_{\mathbb{Z}}} \int_{h(F)} \frac{\phi}{p_\lambda}\frac{dp_1}{p_1}\wedge\cdots\wedge\frac{dp_r}{p_r} z^\lambda, \tag{2.11}$$

and the sum is absolutely convergent on the domain $\psi^{-1}(\mathfrak{a}^*_{[\tau]})$.

Proof. We need two facts from complex function theory. First recall the Laurent expansion of a function of one complex variable. Let ϕ be an holomorphic function on an annulus $\epsilon_1 < |z| < \epsilon_2$. Then, for $\epsilon_1 < \epsilon, |q| < \epsilon_2$, we have

$$\phi(q) = \frac{1}{2\pi\sqrt{-1}} \sum_{n\in\mathbb{Z}} \int_{|y|=\epsilon} \frac{\phi(y)}{y^n}\frac{dy}{y} q^n, \tag{2.12}$$

and the series is uniformly and absolutely convergent on any compact subset of $\epsilon_1 < |q| < \epsilon_2$. A similar statement holds for functions of r complex variables, which are defined on a product on annuli.

The other fact is contained in the following

Lemma 2.9. *Let $\tilde{p}: U \to V$ be a proper holomorphic map between open subsets of a complex vector space, and let $\phi: U \to \mathbb{C}$ be a holomorphic function. Then the push-forward function $\tilde{p}_*\phi$ given by*

$$\tilde{p}_*\phi(v) = \sum_{p(u)=v} \phi(u)$$

is holomorphic on V.

The proof will be omitted (cf [10, Chapter 5]). We will, in fact, apply this to our map p in a domain where it is proper and *non-singular*; in this case the statement is trivial.

Now we are ready to prove the Proposition. Let $\xi \in \mathfrak{a}^*$, and orient the torus

$$\widehat{T}(\xi) = \{q \in (\mathbb{C}^*)^r;\ |q_j| = e^{-\langle \xi, \lambda_j \rangle},\ j = 1, \ldots, r\} \subset (\mathbb{C}^*)^r$$

in the standard fashion. According to Corollary 2.5 (1), if $\xi \in \mathfrak{a}^*_{[\tau]}$, then the map $p_{\boldsymbol{\lambda}} = (p_1, \ldots, p_r)$, even when restricted to $U(F, N)$, is proper to a neighborhood of $\widehat{T}(\xi)$.

Now set $\xi = \psi(z)$, and consider the function ϕ given in the proposition. From the discussion above, we can conclude that the function $p_{\boldsymbol{\lambda}*}\phi$ is holomorphic in a neighborhood of the torus $\widehat{T}(\xi)$, and thus we can write down the standard Laurent expansion for it:

$$\sum_{n \in \mathbb{Z}^r} \frac{1}{(2\pi\sqrt{-1})^r} \int_{\widehat{T}(\xi)} \frac{p_{\boldsymbol{\lambda}*}\phi}{y_1^{n_1} \cdots y_r^{n_r}} \frac{dy_1}{y_1} \frac{dy_2}{y_2} \cdots \frac{dy_r}{y_r} q_1^{n_1} \cdots q_r^{n_r}.$$

Now we pull back this equality by the map $p_{\boldsymbol{\lambda}}$, restricted to $U(F, N)$. For the left hand side we have

$$p_{\boldsymbol{\lambda}*}\phi(q_{\boldsymbol{\lambda}}(z)) = \sum \phi(u),\ u \in O(z, \mathfrak{A}) \cap U(F, N).$$

To compute the pull-back of the right hand side, observe that $p_{\boldsymbol{\lambda}}^{-1}(\widehat{T}(\xi)) = Z(\xi)$, thus using Theorem 2.10 we can conclude that this pull-back equals

$$\sum_{n \in \mathbb{Z}^r} \frac{\nu(F)}{(2\pi\sqrt{-1})^r} \int_{h(F)} \frac{\phi}{p_1^{n_1} \cdots p_r^{n_r}} \frac{dp_1}{p_1} \frac{dp_2}{p_2} \cdots \frac{dp_r}{p_r} q_1^{n_1} \cdots q_r^{n_r}.$$

Thus we recovered the two sides of (2.11), and this completes the proof of the proposition. □

Combining Proposition 2.8 with Lemma 2.1, Corollary 2.5 and Theorem 2.6, we obtain

Corollary 2.10. *Assume that (τ, N) is a sufficient pair of constants. Let $z \in (\mathbb{C}^*)^n$ be such that $\psi(z)$ is $\mathcal{FL}(\mathfrak{A})_\tau$-regular. If K is a holomorphic function defined on $\cup_{F \in \mathcal{FL}(\mathfrak{A}, \psi(z))} U(F, N)$, then we have:*

$$\sum_{u \in O(z, \mathfrak{A})} \frac{K(u)}{G(u)} = \frac{1}{(2\pi\sqrt{-1})^r} \sum_{F \in \mathcal{FL}(\mathfrak{A}, \xi)} \nu(F) \sum_{\lambda \in \mathfrak{a}_{\mathbb{Z}}} \int_{h(F)} \frac{K\, d\mathrm{vol}}{p_\lambda \prod_{i=1}^n \alpha_i} z^\lambda,$$

where G is defined in (1.12)*. The series is absolutely convergent on* $\psi^{-1}(\mathfrak{a}^*_{[\tau]})$.

2.2. **Jeffrey-Kirwan residue and iterated residues.** Consider our exact sequence

$$0 \to \mathfrak{a} \to \mathfrak{g} \xrightarrow{\pi} \mathfrak{t} \to 0 \tag{2.13}$$

and the dual sequence

$$0 \to \mathfrak{t}^* \to \mathfrak{g}^* \xrightarrow{\rho} \mathfrak{a}^* \to 0 \tag{2.14}$$

which is is also exact.

The sequence $\mathcal{B}$ is the image under π of the canonical basis $\{\omega_i\}_{i=1}^n$ of $\mathfrak{g}$, while the sequence $\mathcal{A}$ is the image under ρ of the dual canonical basis $\{\omega^i\}_{i=1}^n$ of $\mathfrak{g}^*$. In the toric mirror residue conjecture of Batyrev-Materov [2], two toric varieties make their appearance. The first one is the toric variety associated to the polytope $\Pi^{\mathfrak{B}} \subset \mathfrak{t}$ obtained as the convex hull of $\mathcal{B}$. The other one $V_{\mathfrak{A}}(\mathfrak{c})$ is determined by a chamber $\mathfrak{c}$ of the cone generated by $\mathfrak{A}$ containing $\kappa_{\mathfrak{A}}$ in their closure.

Recall the definition of a chamber. Consider the cone $\mathrm{Cone}(\mathfrak{A}) \subset \mathfrak{a}^*$ generated by the elements of $\mathfrak{A}$. The subset $\mathrm{Cone}_{\mathrm{sing}}(\mathfrak{A})$ is the set of elements in $\mathrm{Cone}(\mathfrak{A})$ which can be written as a positive linear combination of m elements of $\mathfrak{A}$, with $m < r$. Chambers are the open polyhedral cones in $\mathfrak{a}^*$ which are the connected components of $\mathrm{Cone}(\mathfrak{A}) \setminus \mathrm{Cone}_{\mathrm{sing}}(\mathfrak{A})$. An element ξ in $\mathrm{Cone}(\mathfrak{A}) \setminus \mathrm{Cone}_{\mathrm{sing}}(\mathfrak{A})$ will be called $\mathfrak{A}$-regular; it belongs then to a unique chamber $\mathfrak{c}$.

Now we recall the relation between Minkowski sums of polytopes and chambers.

Definition 2.6. Let $\theta \in \mathfrak{a}^*$ and define the *partition polytope*

$$\Pi_\theta = \rho^{-1}(\theta) \bigcap \sum_{i=1}^n \mathbb{R}^{\geq 0}\omega^i,$$

which lies in an affine subspace of $\mathfrak{g}^*$ parallel to $\mathfrak{t}^*$.

The following proposition is well-known; see [14, Lemma 3.4] for a proof.

Proposition 2.11. *Let θ_k, $k = 1, \ldots, l$ be elements in $\mathfrak{a}^*$. Let $\theta = \sum_{k=1}^l \theta_k$. Then the partition polytope Π_θ is the Minkowski sum of the partition polytopes Π_{θ_k} if and only the elements θ_k are in the closure of the same chamber $\mathfrak{c}$.*

A chamber $\mathfrak{c}$ determines a complete regular simplicial fan in $\mathfrak{t}$; the 1-dimensional faces of these fans are among the rays $\mathbb{R}^+\beta_i, \beta_i \in \mathcal{B}$. A cone $\sum_{i\in\nu} \mathbb{R}^+\beta_i \subset \mathfrak{t}$ belongs to the fan determined by $\mathfrak{c}$ if and only if the cone $\sum_{k\notin\nu} \mathbb{R}^+\alpha_k$ contains the chamber $\mathfrak{c}$. Denote the toric variety associated to this fan by $V_{\mathfrak{A}}(\mathfrak{c})$. It is an orbifold and it can also be realized by a quotient construction (cf [15], Section 1.2). In particular, each polynomial function P on $\mathfrak{a}$ leads to a cohomology class $\chi(P)$ on $V_{\mathfrak{A}}(\mathfrak{c})$ via the Chern-Weil map.

Now we recall a formula from [15] for the intersection numbers on $V_{\mathfrak{A}}(\mathfrak{c})$ for any chamber $\mathfrak{c}$. At the end of this section, the special chambers containing $\kappa_{\mathfrak{A}}$ in their closure will appear.

Associated to each chamber $\mathfrak{c}$, there is a linear form $\phi \mapsto \mathrm{JK}_{\mathfrak{c}}(\phi)$ on $\mathbb{C}_{\mathfrak{A}}[\mathfrak{a}]$ called the *Jeffrey-Kirwan residue.* This linear form vanishes on homogeneous elements of $\mathbb{C}_{\mathfrak{A}}[\mathfrak{a}]$ unless its homogeneous degree is equal to $-r$. We refer to [4] for the definition. Here we only recall an important observation to be used later:

Let $\bar{\mathfrak{c}}^{\perp} \subset \mathfrak{a}$ be the polar cone of the chamber $\mathfrak{c}$:

$$\bar{\mathfrak{c}}^{\perp} = \{\lambda \in \mathfrak{a};\ \langle \xi, \lambda\rangle \geq 0,\ \text{for all } \xi \in \mathfrak{c}\}.$$

Lemma 2.12. [15, Proposition 3.1] *Let $\lambda \in \mathfrak{a}_Z$ such that λ does not belong to $\bar{\mathfrak{c}}^{\perp}$, and let P be a polynomial on $\mathfrak{a}$. Then*

$$\mathrm{JK}_{\mathfrak{c}}\left(\frac{P}{p_\lambda \prod_{i=1}^n \alpha_i}\right) = 0$$

As explained in [15], an algebraic formula for integrating a cohomology class over $V_{\mathfrak{A}}(\mathfrak{c})$ can be given in terms of the Jeffrey-Kirwan residue. For a polynomial P on $\mathfrak{a}$:

$$\int_{V_{\mathfrak{A}}(\mathfrak{c})} \chi(P) = JK_{\mathfrak{c}}\left(\frac{P}{\prod_{i=1}^n \alpha_i}\right). \tag{2.15}$$

We recall now our construction of a concrete, compact, real-analytic cycle in $U(\mathfrak{A})$, such that the linear form $\mathrm{JK}_{\mathfrak{c}}(\phi)$ is simply obtained by integration of $\phi(u)\, d\mathrm{vol}$ on this cycle.

Definition 2.7. For a flag $F \in \mathcal{FL}(\mathfrak{A})$, introduce the acute cone $\mathfrak{s}^+(F, \mathfrak{A})$ generated by the non-negative linear combinations of the elements $\{\kappa_j^F, j = 1, \dots, r\}$

$$\mathfrak{s}^+(F, \mathfrak{A}) = \sum_{j=1}^r \mathbb{R}^{\geq 0} \kappa_j^F.$$

For $\xi \in \mathfrak{a}^*$ denote by $\mathcal{FL}^+(\mathfrak{A}, \xi)$ those flags in $\mathcal{FL}(\mathfrak{A})$ for which $\xi \in \mathfrak{s}^+(F, \mathfrak{A})$.

We say that $\xi \in \mathfrak{a}^*$ is $\mathcal{FL}(\mathfrak{A})^+$*-regular* if ξ is not on the boundary of any of the cones $\mathfrak{s}^+(F, \mathfrak{A})$, $F \in \mathcal{FL}^+(\mathfrak{A}, \xi)$. This means that for any flag $F \in \mathcal{FL}^+(\mathfrak{A}, \xi)$, all the coefficients, $m_1, m_2, \dots, m_r$, in any linear expression

$$\xi = m_1\kappa_1^F + m_2\kappa_2^F + \dots + m_{r-1}\kappa_{r-1}^F + m_r\kappa_r^F$$

are nonzero.

We will write $\xi \in \mathcal{FL}(\mathfrak{A})^+_\tau$ if $m_j > \tau$, $j = 1, \dots, r$, in this expression.

If ξ is a $\mathcal{FL}(\mathfrak{A})^+$-regular element, then any flag F in $\mathcal{FL}^+(\mathfrak{A}, \xi)$ is proper, and the cone $\mathfrak{s}^+(F, \mathfrak{A})$ is a closed simplicial cone.

Lemma 2.13. (1) *Any $\mathcal{FL}(\mathfrak{A})^+$-regular element ξ is $\mathfrak{A}$-regular.*

(2) *If ξ is $\mathfrak{A}$-regular and $\mathcal{FL}(\mathfrak{A})$-regular, then ξ is also $\mathcal{FL}(\mathfrak{A})^+$-regular.*

(3) *If ξ is $\mathcal{FL}(\mathfrak{A})$-regular, then for sufficiently large $s > 0$, the element $\xi + s\kappa_{\mathfrak{A}}$ is a $\mathcal{FL}(\mathfrak{A})^+$-regular element, moreover $\mathcal{FL}(\mathfrak{A}, \xi) = \mathcal{FL}^+(\mathfrak{A}, s\kappa_{\mathfrak{A}} + \xi)$.*

Remark 2.4. Note that an element ξ which is $\mathcal{FL}(\mathfrak{A})$-regular is not usually $\mathfrak{A}$-regular. For example if $\mathfrak{a}^*$ has basis e_1, e_2 and if $\mathfrak{A} := [e_1, e_2]$ the singular element $\xi = e_1$ is $\mathcal{FL}(\mathfrak{A})$-regular.

Proof. First of all, let us prove by induction on the dimension of $\mathfrak{a}^*$ that for any ξ in $\mathrm{Cone}(\mathfrak{A})$ there exists a proper flag F such that $\xi \in \mathfrak{s}^+(F, \mathfrak{A})$. Indeed, the point $\kappa_{\mathfrak{A}} = \kappa_r$ is in the interior of $\mathrm{Cone}(\mathfrak{A})$. Thus there exists a face Φ of $\mathrm{Cone}(\mathfrak{A})$ such that ξ belongs to the convex hull of $\kappa_{\mathfrak{A}}$ and of $\mathrm{Cone}(\mathfrak{A} \cap \Phi)$. Let $t \geq 0$ such that $\xi' = \xi - t\kappa_{\mathfrak{A}}$ belongs to the cone $\mathrm{Cone}(\mathfrak{A} \cap \Phi)$. If F' is a proper flag for the system $\mathfrak{A} \cap \Phi$ such that $\xi' \in \mathfrak{s}^+(F', \mathfrak{A} \cap \Phi)$, then $[F', \mathfrak{a}^*]$ is a proper flag and $\xi \in \mathfrak{s}^+(F, \mathfrak{A})$.

We now prove the first statement. If ξ is $\mathfrak{A}$-singular, then there is a hyperplane F_{r-1} spanned by vectors of $\mathfrak{A}$ such that $\xi \in \mathrm{Cone}(\mathfrak{A} \cap F_{r-1})$. We choose a flag F' relative to the system $\mathfrak{A} \cap F_{r-1}$ such that $\xi \in \mathfrak{s}^+(F', \mathfrak{A} \cap F_{r-1})$. Then $F = [F', \mathfrak{a}^*]$ belongs to $\mathcal{FL}(\mathfrak{A})$, ξ is in $\mathfrak{s}^+(F, \mathfrak{A})$, but the κ_r-coefficient of ξ on $\kappa_{\mathfrak{A}}$ vanishes. This is in contradiction with the assumption that ξ is $\mathcal{FL}(\mathfrak{A})^+$-regular.

To prove the second statement, observe that if ξ is $\mathcal{FL}(\mathfrak{A})$-regular but not $\mathcal{FL}(\mathfrak{A})^+$-regular, then ξ is in a cone spanned by the vectors κ_j^F, $j = 1, \ldots, r-1$, for some flag F. Clearly, these vectors are in the cone spanned by $\mathfrak{A} \cap F_{r-1}$. This implies that ξ is $\mathfrak{A}$-singular.

The third item is clear. □

The following theorem is proved in [15] under a stronger assumption of regularity. However the same proof shows the following.

Theorem 2.14 ([15, Theorem 2.6])**.** *Let $\mathfrak{c}$ be any chamber of the projective sequence $\mathfrak{A}$, and let ξ be a vector in $\mathfrak{c}$ which is $\mathcal{FL}(\mathfrak{A})^+$-regular. Then for every $\phi \in \mathbb{C}_{\mathfrak{A}}[\mathfrak{a}]$*

$$\mathrm{JK}_{\mathfrak{c}}(\phi) = \sum_{F \in \mathcal{FL}^+(\mathfrak{A}, \xi)} \nu(F) \operatorname*{Res}_F \phi. \tag{2.16}$$

Consider now a $\mathcal{FL}(\mathfrak{A})$-regular vector ξ in $\mathfrak{a}^*$. It follows from Lemma 2.13 that for sufficiently large real s, the element $s\kappa_{\mathfrak{A}} + \xi$ is $\mathfrak{A}$-regular. We can thus associate to every $\mathcal{FL}(\mathfrak{A})$-regular element ξ a chamber $\mathfrak{c}(\kappa_{\mathfrak{A}}, \xi)$ as the unique chamber containing $s\kappa_{\mathfrak{A}} + \xi$ for sufficiently large s. Clearly, when $\kappa_{\mathfrak{A}}$ is not in $\mathrm{Cone}_{\mathrm{sing}}(\mathfrak{A})$, then the chamber $\mathfrak{c}(\kappa_{\mathfrak{A}}, \xi)$ is the chamber which contains $\kappa_{\mathfrak{A}}$. In general the chamber $\mathfrak{c}(\kappa_{\mathfrak{A}}, \xi)$ is a chamber containing $\kappa_{\mathfrak{A}}$ in its closure.

Lemma 2.13 and Theorem 2.14 have the following corollary:

Corollary 2.15. *Let $\phi \in \mathbb{C}_{\mathfrak{A}}[\mathfrak{a}]$ and let ξ be $\mathcal{FL}(\mathfrak{A})$-regular. Then*

$$\mathrm{JK}_{\mathfrak{c}(\kappa_{\mathfrak{A}}, \xi)}(\phi) = \sum_{F \in \mathcal{FL}(\mathfrak{A}, \xi)} \nu(F) \operatorname*{Res}_F(\phi).$$

The following key statement easily follows from Lemma 2.13, and the discussion afterwards.

Proposition 2.16 ([15, Proposition 6.1]). *Let $\mathfrak{c}$ be a chamber such that $\kappa_{\mathfrak{A}}$ is in the closure of $\mathfrak{c}$, and let $\xi \in \mathfrak{c}$ be a $\mathcal{FL}(\mathfrak{A})$-regular element. Then $\mathcal{FL}(\mathfrak{A}, \xi) = \mathcal{FL}^+(\mathfrak{A}, \xi)$.*

This allows us to formulate our central result in a concise fashion as follows:

Corollary 2.17. *Let $\mathfrak{A} = [\alpha_1, \dots, \alpha_n]$ be a projective sequence in $\mathfrak{a}^*$, and $\mathfrak{c}$ be a chamber of $\mathfrak{A}$ such that $\kappa_{\mathfrak{A}} = \sum_{i=1}^n \alpha_i \in \bar{\mathfrak{c}}$. Then for any polynomial Q on $\mathfrak{a}$ we have*

$$\int_{V_{\mathfrak{A}}(\mathfrak{c})} \chi(Q) = \int_{Z(\xi)} \frac{Q\, d\mathrm{vol}}{\prod_{i=1}^n \alpha_i},$$

where $V_{\mathfrak{A}}(\mathfrak{c})$ is the orbifold toric variety associated to $\mathfrak{A}$ and $\mathfrak{c}$, $\chi(Q)$ is the class in $H^*(V_{\mathfrak{A}}(\mathfrak{c}), \mathbb{R})$ corresponding to Q under the Chern-Weil map, ξ is a $\mathcal{FL}(\mathfrak{A})_\tau$-regular vector in $\mathfrak{c}$ and the cycle $Z(\xi)$ is defined in (2.6)

3. The mixed mirror residue conjecture

We start with our usual data:

$$0 \to \underline{\mathfrak{a}} \xrightarrow{\iota} \underline{\mathfrak{g}} \xrightarrow{\pi} \underline{\mathfrak{t}} \to 0$$

an exact sequence of real vector spaces of dimensions $\underline{r}, \underline{n}, \underline{d}$ respectively; each vector space is endowed with an integral structure, and the sequence is exact over the integers. The vector space $\underline{\mathfrak{g}}$ is endowed with a basis $\{\omega_i, 1 \le i \le \underline{n}\}$, so that

$$\underline{\mathfrak{g}} = \oplus_{i=1}^{\underline{n}} \mathbb{R}\omega_i, \qquad \underline{\mathfrak{g}}_{\mathbb{Z}} = \oplus_{i=1}^{\underline{n}} \mathbb{Z}\omega_i.$$

Let $\underline{\mathfrak{B}} = [\underline{\beta}_1, \underline{\beta}_2, \dots, \underline{\beta}_{\underline{n}}]$ be the sequence of vectors in $\underline{\mathfrak{t}}_{\mathbb{Z}}$ which are images of the basis vectors under $\underline{\pi}$: $\underline{\beta}_i = \underline{\pi}(\omega_i)$. We assume that 0 is in the interior of the convex hull $\Pi^{\underline{\mathfrak{B}}}$ of the elements of $\underline{\mathfrak{B}}$, and that $\underline{\mathfrak{B}}$ generates $\mathfrak{t}_{\mathbb{Z}}$ over $\mathbb{Z}$. We denote by $\underline{\mathfrak{A}} = [\underline{\alpha}_1, \underline{\alpha}_2, \dots, \underline{\alpha}_{\underline{n}}]$ the Gale dual sequence in $\underline{\mathfrak{a}}^*$, i.e. the sequence obtained as the restrictions to $\underline{\mathfrak{a}}$ of the coordinate functions $\omega^i \in \underline{\mathfrak{g}}^*$ $(1 \le i \le \underline{n})$. Then $\underline{\mathfrak{A}}$ generates $\underline{\mathfrak{a}}^*$. We introduce the notation $\underline{\kappa}$ for the vector $\kappa_{\underline{\mathfrak{A}}} = \sum_{i=1}^{\underline{n}} \underline{\alpha}_i$.

Following [3], consider a decomposition of the set $\{1, 2, \dots, \underline{n}\}$ into l disjoint sets:

$$\{1, 2, \dots, \underline{n}\} = \bigcup_{k=1}^{l} D_k, \tag{3.1}$$

and define $\theta_k = \sum_{i \in D_k} \underline{\alpha}_i$. Note that $\underline{\kappa} = \sum_{k=1}^{l} \theta_k$.

Given this data, we construct an example of the setup described at the beginning of the paper.

We set $V = \underline{\mathfrak{t}} \oplus \mathbb{R}^l$; let $[\gamma_1, \dots, \gamma_l]$ be the canonical basis of $\mathbb{R}^l$, so that

$$V = \underline{\mathfrak{t}} \oplus \bigoplus_{k=1}^{l} \mathbb{R}\gamma_k.$$

Then V is a $d+1 = \underline{d} + l$ dimensional vector space endowed with a lattice.

We choose the grading vector $g \in V^*$ to be the linear functional which vanishes on $\underline{\mathfrak{t}}$ and takes the value 1 on the vectors γ_k, $k = 1, \dots, l$. The set $\mathfrak{M}$ is constructed as follows: for each $\underline{\beta}_i \in \underline{\mathfrak{B}}$, let $\mu_i = \underline{\beta}_i + \gamma_k \in V$, where k is the integer satisfying $i \in D_k$, and set

$$\mu_i = \gamma_{i-\underline{n}}, \quad i = \underline{n}+1, \dots, \underline{n}+l.$$

Thus our sequence $\mathfrak{M} = [\mu_1, \dots, \mu_n]$ contains $n = \underline{n}+l$ elements, and further defines an exact sequence

$$0 \to W \to \underline{\mathfrak{g}} \oplus \mathbb{R}^l \to \underline{\mathfrak{t}} \oplus \mathbb{R}^l \to 0,$$

where the map from $\underline{\mathfrak{g}} \oplus \mathbb{R}^l$ to $\underline{\mathfrak{t}} \oplus \mathbb{R}^l$ sends ω_i with $i \leq \underline{n}$ to $\mu_i = \underline{\beta}_i + \gamma_k$ and γ_k to γ_k. Comparing this with the diagram (1.3), we see that we have $\mathfrak{g} = \underline{\mathfrak{g}} \oplus \mathbb{R}^l$, and $W = \underline{\mathfrak{a}}$.

We can identify the space $W \subset \underline{\mathfrak{g}} \oplus \mathbb{R}^l$ with $\underline{\mathfrak{a}}$ via the map $u \mapsto u - \sum_{k=1}^{l} \langle \theta_k, u \rangle \gamma_k$. Indeed, the image of the element $u = \sum_{i=1}^{\underline{n}} \underline{\alpha}_i(u)\omega_i \in \underline{\mathfrak{g}} \oplus \mathbb{R}^l$ in $\underline{\mathfrak{t}} \oplus \mathbb{R}^l$ is $\sum_{i=1}^{\underline{n}} \underline{\alpha}_i(u)(\underline{\beta}_i + \gamma_k) = \sum_{k=1}^{l} \langle \theta_k, u \rangle \gamma_k$.

We observe that the cone of $C = C(\mathfrak{M})$ defined in §1.1 is exactly the construction of the Cayley cone given in [3, §3]. Denote by Δ_k the convex hull of $\{0\}$ and the elements β_i with $i \in D_k$. Then the base of the cone $C = C(\mathfrak{M})$ is the convex hull of the union

$$\Delta_1 \times \{\gamma_1\} \cup \cdots \cup \Delta_l \times \{\gamma_l\} \subset \mathfrak{g},$$

and our cone $C(\mathfrak{M})$ is the cone $C(\Delta_1, \Delta_2, \dots, \Delta_r)$ in [3, Definition 3.1].

Implementing the construction in §1.1 further, we see that in our present setup there is a natural candidate for the vector $\gamma \in V$:

$$\gamma = \sum_{k=1}^{l} \gamma_k. \tag{3.2}$$

This vector indeed lies in the interior of the cone C generated by $\mathfrak{M}$; it has degree l.

Now we define the vector space $\mathfrak{t} = V/\mathbb{R}\gamma$, which is of dimension $d = \underline{d} + l - 1$, and we consider the map $\pi : \mathfrak{g} \to \mathfrak{t}$, which sends ω_i to μ_i modulo γ. The images of the basis vectors will again be denoted by βs, thus we have

$$\mathfrak{B} = [\beta_i = \pi(\mu_i),\ i = 1, \dots, n].$$

Note that $\pi(\gamma) = 0$ and that the space $\mathfrak{a}$, the kernel of π, may be canonically identified with $\underline{\mathfrak{a}} \oplus \mathbb{R}\gamma$. This decomposition holds over the integers as well.

Now denote by t the linear functional on $\mathfrak{a}$ which is zero on $\underline{\mathfrak{a}}$ and satisfies $\langle t,\gamma\rangle = 1$. Then we have a canonical decomposition $\mathfrak{a}^* = \underline{\mathfrak{a}}^* \oplus \mathbb{R}t$. Again, we compute the Gale duals, and we obtain

Lemma 3.1. *The Gale dual sequence* $\mathfrak{A}$ *of the sequence* $\mathfrak{B}$ *is*

$$\underline{\alpha}_i,\ i = 1,\dots,\underline{n};\ \alpha_i = t-\theta_{i-\underline{n}},\ i = \underline{n}+1,\dots,n.$$

Thus

$$\mathfrak{A} = [\underline{\alpha}_1,\underline{\alpha}_2,\dots,\underline{\alpha}_{\underline{n}}, t-\theta_1, t-\theta_2,\dots,t-\theta_l].$$

Note that since $\sum_{k=1}^{l}\theta_k = \sum_{i=1}^{n}\underline{\alpha}_i$, we have $\kappa_{\mathfrak{A}} = lt$.

Consider the partition polytope $\Pi_{\underline{\kappa}} \subset \underline{\mathfrak{g}}^*$ and the point $b = \sum_{i=1}^{n}\omega^i \in \Pi_{\underline{\kappa}}$. Define the polytope $\underline{\Pi}$ as $\Pi_{\underline{\kappa}} - b$. An easy computation shows that the dual polytope of $\underline{\Pi}$ is the convex hull of the set $\underline{\mathfrak{B}}$. Consider the partition polytopes Π_{θ_k}, and define $\underline{\Pi}_k = \Pi_{\theta_k} - b_k \subset \underline{\mathfrak{t}}^*$, where $b_k = \sum_{i\in D_k}\omega^i \in \Pi_{\theta_k}$.

The following proposition follows directly from Proposition 2.11.

Proposition 3.2. *The polytope* $\underline{\Pi}$ *is the Minkowski sum of the polytopes* $\underline{\Pi}_k$ *if and only if the elements* θ_k *are in the closure of the same chamber* $\mathfrak{c}\subset\underline{\mathfrak{a}}^*$.

Definition 3.1. Let $\mathfrak{c}$ be a chamber of the Cone($\underline{\mathfrak{A}}$). We will say that the projective sequence $\underline{\mathfrak{A}}$ is $\mathfrak{c}$-partitioned into l parts if the chamber $\mathfrak{c}$ contains in its closure all the elements $\theta_k = \sum_{i\in D_k}\underline{\alpha}_i$, $k = 1,\dots,l$.

Note that in this case the chamber $\mathfrak{c}$ also contains in its closure the element $\underline{\kappa} = \kappa_{\underline{\mathfrak{A}}} = \sum_{k=1}^{l}\theta_k$.

Consider (cf. [3, Definition 4.2]) the dual family $\overline{\Pi}_1,\dots,\overline{\Pi}_l$ of the family of polytopes $\underline{\Pi}_k$ defined by the equations

$$\overline{\Pi}_k = \{y\in\underline{\mathfrak{t}};\langle x,y\rangle \geq \delta_{ks}, x\in\underline{\Pi}_s, s=1,\dots,l\}.$$

Proposition 3.3. *(* [3, Definition 4.2]*) Assume* $\underline{\mathfrak{A}}$ *is* $\mathfrak{c}$*-partitioned into* l *parts. Then the polytope* $\overline{\Pi}_k$ *is the convex hull of the origin* $\{0\}$ *and the vectors* $\underline{\beta}_i$ *with* $i\in D_k$.

Proof. First we verify that the points $\underline{\beta}_i$ with $i\in D_k$ are in $\overline{\Pi}_k$. Let $c\in\underline{\Pi}_s\subset\underline{\mathfrak{t}}^*$. Then c may be written as $\sum_{i=1}^{\underline{n}} c_i\omega^i - \sum_{i\in D_k}\omega^i$, with $c_i\geq 0$. Since $\beta_i = \pi(\omega_i)$, the value of c on β_i is $\langle c,\omega_i\rangle$; this, in turn, equals $c_i - 1$ if $i\in D_k$, and c_i if $i\notin D_k$. Thus β_i with $i\in D_k$ satisfies the inequalities defining $\overline{\Pi}_k$.

Conversely, let $y\in\overline{\Pi}_k$, with $y\neq 0$, and choose a cone of the fan in $\underline{\mathfrak{t}}$ determined by $\mathfrak{c}$, which contains y. This means that there is a $\underline{d}$-element subset $\nu\subset\{1,\dots,\underline{n}\}$ such that $y = \sum_{i\in\nu}t_i\beta_i$ with $t_i\geq 0$, and $\mathfrak{c}\subset\sum_{i\notin\nu}\mathbb{R}^+\alpha_i$. Since we have a $\mathfrak{c}$-partition, this implies that we can write $\theta_s = \sum_{j\notin\nu}x_s^j\alpha_j$, with $x_s^j\geq 0$; this means that the point $\rho_s = \sum_{j\notin\nu}x_s^j\omega^j - \sum_{i\in D_s}\omega^i$ is in $\underline{\Pi}_s$ for $1\leq s\leq l$. By the definition of $\overline{\Pi}_k$, we have $\langle\rho_s,y\rangle\geq 0$ for $s\neq k$. Since $\langle\rho_s,y\rangle\geq 0 = \sum_{i\in D_s\cap\nu}t_a$, it follows that ν is contained in D_k. We also have $\langle\rho_k,y\rangle\geq -1$, hence $-\sum_{i\in\nu}t_i\geq -1$, thus $y = \sum_{a\in\nu}t_i\beta_i + (1-\sum_{i\in\nu}t_i)0$ is in the convex hull of the elements $\{\beta_i,\ i\in D_k\}$ and 0. $\square$

The following proposition will be crucial in the proof of the mixed toric residue mirror conjecture.

Proposition 3.4. *Assume $\underline{\mathfrak{A}}$ is $\mathfrak{c}$-partitioned into l parts, i.e. $\theta_k \in \bar{\mathfrak{c}}$, $k = 1, \dots, l$. Let $\underline{\xi} \in \mathfrak{c} \subset \underline{\mathfrak{a}}^*$ and $\xi \in \mathfrak{a}^*$ be such that $\xi - \underline{\xi} = bt$ for some $b \in \mathbb{R}$. Then the map $\underline{F} \mapsto (\underline{F}, \mathfrak{a}^*)$ from $\mathcal{FL}(\underline{\mathfrak{A}})$ to $\mathcal{FL}(\mathfrak{A})$ induces a bijection between $\mathcal{FL}(\underline{\mathfrak{A}}, \underline{\xi})$ and $\mathcal{FL}(\mathfrak{A}, \xi)$.*

Proof. The injectivity of the correspondence $\underline{F} \mapsto (\underline{F}, \mathfrak{a}^*)$ is obvious, thus we only need to show surjectivity. Let

$$F = [F_0 = \{0\} \subset F_1 \subset F_2 \subset \cdots \subset F_{r-1} \subset F_r \subset F_{r+1} = \underline{\mathfrak{a}}^* \oplus \mathbb{R}t] \in \mathcal{FL}(\mathfrak{A}, \xi).$$

By definition of $\mathcal{FL}(\mathfrak{A}, \xi)$, we can write $\xi = \sum_{j=1}^{r+1} c_j \kappa_j^F$ with $c_j \geq 0$ for $j \leq r$. Using the description of $\mathfrak{A}$ in Lemma 3.1 and projecting this equality onto $\underline{\mathfrak{a}}^*$ along $\mathbb{R}t$ we obtain

$$\underline{\xi} + \sum_{j=1}^{r} c_j \sum_{\{k;\, t-\theta_k \in F_j\}} \theta_k = \sum_{j=1}^{r} c_j \sum_{\{i;\, \underline{\alpha}_i \in F_j\}} \underline{\alpha}_i. \tag{3.3}$$

Now assume that $F_r \neq \underline{\mathfrak{a}}^*$. Then the elements $\underline{\alpha}_i$ on the right hand side do not span $\underline{\mathfrak{a}}^*$, and, therefore, the right hand side lies in $\mathrm{Cone}_{\mathrm{sing}}(\underline{\mathfrak{A}})$. On the other hand, according to our assumptions on the θ_ks and $\underline{\xi}$, the left hand side of (3.3) is in $\mathfrak{c}$. This is a contradiction, since a vector is in $\mathrm{Cone}_{\mathrm{sing}}(\underline{\mathfrak{A}})$ exactly when it is not in a chamber. Thus we have shown that any flag $F \in \mathcal{FL}(\mathfrak{A}, \xi)$ is of the form $F = (\underline{F}, \mathfrak{a}^*)$, where $\underline{F}$ is an element of $\mathcal{FL}(\underline{\mathfrak{A}}, \underline{\xi})$. □

Corollary 3.5. *With the assumptions and notations of the Proposition, the vector $\underline{\xi} \in \mathfrak{c} \subset \underline{\mathfrak{a}}^*$ is $\mathcal{FL}(\underline{\mathfrak{A}})^+$-regular ($\mathcal{FL}(\underline{\mathfrak{A}})^+_\tau$-regular) if and only if the vector $\underline{\xi} + bt \in \mathfrak{a}^*$ is $\mathcal{FL}(\mathfrak{A})$-regular ($\mathcal{FL}(\mathfrak{A})_\tau$-regular).*

Now let $z = (z_1, z_2, \dots, z_{\underline{n}})$ be a vector in $\mathbb{C}^{\underline{n}}$, and introduce the vector $\tilde{z} = (z_1, z_2, \dots, z_{\underline{n}}, 1, 1, \dots, 1) \in \mathbb{C}^n$.

Then, as in §1.2, we consider

$$f_{\tilde{z}} = \sum_{k=1}^{l} e_{\gamma_k} + \sum_{i=1}^{\underline{n}} z_i e_{\mu_i} \in S^1(C).$$

This function may be written as

$$f_{\tilde{z}} = \sum_{k=1}^{l} e_{\gamma_k} (1 + \sum_{i \in D_k} z_i e_{\underline{\beta}_i}).$$

Note that after the substitution $z_i = -a_i$, our function $f_{\tilde{z}}$ is the Cayley polynomial F associated to the functions f_k and the sets $A_k = \{\beta_i; i \in D_k\}$ in [2, §4].

As in §1.2, set $f_{\tilde{z},0} = f_{\tilde{z}}$ and consider the sequence $\{f_{\tilde{z},1}, \dots, f_{\tilde{z},d}\}$ of derivatives of the function $f_{\tilde{z}}$, with respect to a basis $\{a_1, a_2, \dots, a_d\}$ of $\mathfrak{t}^*$. Comparing this setup to §1.2 on toric residue, we see that the function $f_{\tilde{z}}$ is

the function associated to the sequence $\mathfrak{M}$ and the *non-generic* parameter $\tilde{z}$. However, we have the following lemma.

Lemma 3.6. *For generic $z \in \mathbb{C}^{\underline{n}}$, the sequence $\mathfrak{M}$, the vector γ given in* (3.2) *and the parameter $\tilde{z}$ satisfy conditions* (1.4) *and* (1.5)*. This means that the sections $\{f_{\tilde{z},0}, f_{\tilde{z},1}, \ldots, f_{\tilde{z},d}\}$ have no common zeroes in* $\mathrm{Tvar}_g(C)$*. Furthermore, the common zeroes of the sections $\{f_{\tilde{z},1}, \ldots, f_{\tilde{z},d}\}$ are in the torus* $\mathrm{T}_{\mathbb{C}}(H^*) \subset \mathrm{Tvar}_g(C)$.

Proof. This is a consequence of the results of the Appendix. Indeed consider the subset $\sigma \subset \{1, 2, \ldots, n\}$ consisting of $\underline{n}+1, \ldots, n = \underline{n}+l$. Then $(\mathbb{C}^*)^{\underline{n}}$ may be identified with the set Z_σ of Definition 4.2 by associating to the vector $z = (z_1, \ldots, z_{\underline{n}})$ the vector $\tilde{z} = (z_1, \ldots, z_{\underline{n}}, 1, \ldots, 1)$. The corresponding μ_i with $i \in \sigma$ are the set of l vectors γ_k, $k = 1, \ldots, l$. These are linearly independent by definition, and thus the statement of the Lemma follows from Propositions 4.5 and 4.6 of the Appendix. □

Thus for generic z in $\mathbb{C}^{\underline{n}}$ the conditions (1.4) and (1.5) are satisfied and the toric residue with respect to $\mathbf{f}(\tilde{z})$ is well defined. Moreover, the conditions of Proposition 1.5 are also satisfied.

Now we are ready to formulate our main result: a generalization of the mixed mirror residue conjecture of Batyrev and Materov. Fix a polynomial P of degree $\underline{d}$ in $\underline{n}$ variables. Then

$$e_\gamma P(z_1 e_{\mu_1}, \ldots, z_{\underline{n}} e_{\mu_{\underline{n}}}) \tag{3.4}$$

is an element of $I^{d+1}(C)$. The toric residue of this element with respect to the sequence $\mathbf{f}(\tilde{z})$ is well-defined; it is a rational function of z. The conjecture expresses this toric residue as a sum of intersection numbers on a sequence of Morrison-Plesser moduli spaces.

Assume that we have a $\mathfrak{c}$-partition of $\underline{\mathfrak{A}}$, i.e. that all the elements $\theta_k \in \underline{\mathfrak{a}}^*$, $k = 1, \ldots, l$ belong to the closure of the same chamber $\mathfrak{c}$ of the projective sequence $\underline{\mathfrak{A}}$. Consider the polar cone $\bar{\mathfrak{c}}^\perp$

$$\bar{\mathfrak{c}}^\perp = \{\lambda \in \underline{\mathfrak{a}};\ \langle \alpha, \lambda \rangle \geq 0 \text{ for all } \alpha \in \mathfrak{c}\}.$$

If $\lambda \in \bar{\mathfrak{c}}^\perp$, then the numbers $\langle \theta_k, \lambda \rangle$ are non-negative integers. In particular,

$$\frac{P(\underline{\alpha}_1, \ldots, \underline{\alpha}_n) \prod_{k=1}^{l} \theta_k^{\langle \theta_k, \lambda \rangle}}{\prod_{i=1}^{n} \alpha_i^{\langle \alpha_i, \lambda \rangle + 1}}$$

is an element of degree $-r$ of $\mathbb{C}_{\underline{\mathfrak{A}}}[\underline{\mathfrak{a}}]$, and we can take its Jeffrey-Kirwan residue with respect to the chamber $\mathfrak{c}$.

Theorem 3.7. *Let $\mathfrak{c}$ be a chamber of the projective sequence $\underline{\mathfrak{A}}$, and assume that $\underline{\mathfrak{A}}$ is $\mathfrak{c}$-partitioned into l parts. Let $C = C(\mathfrak{M})$ be the corresponding Cayley cone and let $f_{\tilde{z}} = \sum_{k=1}^{l} e_{\gamma_k}(1 + \sum_{i=1}^{\underline{n}} z_i e_{\underline{\beta}_i})$ be the Cayley polynomial associated to the partition. Denote by $\mathbf{f}(\tilde{z})$ the associated sequence of partial derivatives of $f_t z$.*

Choose a vector $z \in \mathbb{C}^{\underline{n}}$ such that $\psi(z) = \psi_{\underline{\mathfrak{A}}}(z)$ is a $\mathcal{FL}(\underline{\mathfrak{A}})_\tau$-regular element of $\mathfrak{c}$ for some sufficiently large τ. Let P be a polynomial of degree $\underline{d}$ in $\underline{n}$ variables. Then the series

$$\sum_{\lambda \in \bar{\mathfrak{c}}^\perp} \mathrm{JK}_{\mathfrak{c}} \left(\frac{P(\underline{\alpha}_1, \dots, \underline{\alpha}_{\underline{n}}) \prod_{k=1}^{l} \theta_k^{\langle \theta_k, \lambda \rangle}}{\prod_{i=1}^{\underline{n}} \underline{\alpha}_i^{\langle \underline{\alpha}_i, \lambda \rangle + 1}} \right) z^\lambda \tag{3.5}$$

converges absolutely, and sums to the toric residue

$$\mathrm{TorRes}_{\mathbf{f}(\tilde{z})} e_\gamma P(z_1 e_{\mu_1}, \dots, z_{\underline{n}} e_{\mu_{\underline{n}}}) \tag{3.6}$$

Remark 3.1.

- In [3], Batyrev and Materov requires that all polytopes $\underline{\Pi}_k$ have integral vertices. We do not require this hypothesis.
- If the polynomial P is supposed to be multi-homogeneous of degree $\underline{d}_k$ with respect to the variables x_i with $i \in D_k$ with $\sum_{k=1}^{l} \underline{d}_k = \underline{d}$, then

  $$P(z_1 e_{\mu_1}, \dots, z_{\underline{n}} e_{\mu_{\underline{n}}}) = e_{\underline{d}_1 \gamma_1} \cdots e_{\underline{d}_k \gamma_k} P(z_1 e_{\underline{\beta}_1}, \dots, z_{\underline{n}} e_{\underline{\beta}_{\underline{n}}})$$

  and $e_{\underline{d}_1 \gamma_1} \cdots e_{\underline{d}_k \gamma_k} P(z_1 e_{\beta_1}, \dots, z_{\underline{n}} e_{\beta_{\underline{n}}})$ is an element of $S^{\underline{d}}(C)$. In the situation considered in [3], the Cayley cone C is a reflexive Gorenstein cone and $I^{\underline{d}+l}(C)$ is isomorphic to $S^{\underline{d}}(C)$ by multiplication by e_γ.
- As shown in [15] the number $I(P, \beta)$ of [3] is equal to

  $$\mathrm{JK}_{\mathfrak{c}} \left(\frac{P(\underline{\alpha}_1, \dots, \underline{\alpha}_{\underline{n}}) \prod_{k=1}^{l} \theta_k^{\langle \theta_k, \lambda \rangle}}{\prod_{i=1}^{\underline{n}} \underline{\alpha}_i^{\langle \underline{\alpha}_i, \lambda \rangle + 1}} \right)$$

Proof. We showed in Lemma 3.6 that the conditions of the Proposition 1.5 hold in the case of the toric residue (3.6). As a result we can apply the formula (1.13).

As explained in section §1.3, we can parameterize the set $\widetilde{\mathrm{Crit}}(z)$ by elements

$$(u, t) \in O(z, \mathfrak{A}) \subset \underline{\mathfrak{a}}_{\mathbb{C}} \oplus \mathbb{C}\gamma.$$

Note that the character e_γ is equal to 1 on $O(z, \mathfrak{A})$, thus for $(u, t) \in O(z, \mathfrak{A})$, we have $\prod_{k=1}^{l} (t - \theta_k(u)) = 1$.

Recall that $\kappa(\mathfrak{A}) = lt$. Then the toric residue (3.6) may be written as

$$l \sum_{(u,t) \in O(z, \mathfrak{A})} \frac{P(\underline{\alpha}_1(u), \dots, \underline{\alpha}_{\underline{n}}(u))}{ltG(u, t)},$$

where $G(u, t)$ is given by (1.12).

Recall that we associated the vector $\tilde{z} = (z_1, \dots, z_{\underline{n}}, 1, \dots, 1) \in \mathbb{C}^n$ to $z \in \mathbb{C}^{\underline{n}}$. Then $\psi_{\mathfrak{A}}(\tilde{z}) = (\psi_{\underline{\mathfrak{A}}}(z), 0) \in \mathfrak{a}^* = \underline{\mathfrak{a}}^* \oplus \mathbb{C}t$. Recall from Corollary 3.5 that the vector $\psi_{\mathfrak{A}}(\tilde{z}) \in \mathfrak{a}^*$ is $\mathcal{FL}(\mathfrak{A})_\tau$-regular if and only if $\psi_{\underline{\mathfrak{A}}}(z) \in \underline{\mathfrak{a}}^*$ is $\mathcal{FL}^+(\underline{\mathfrak{A}})$- τ-regular. We choose τ large enough in order that Theorem 2.10 holds for $\mathfrak{A}$.

Then using Lemma 3.1 and Theorem 2.10 we can conclude that the toric residue (3.6) is equal to

$$\sum_{F\in\mathcal{FL}(\mathfrak{A},\xi)}\nu(F)\sum_{\Lambda\in\mathfrak{a}_{\mathbb{Z}}}\operatorname*{Res}_F\left(\frac{K(u,t)}{p_\Lambda(u,t)}\frac{1}{\prod_{i=1}^{n}\underline{\alpha}_i(u)\prod_{k=1}^{l}(t-\theta_k(u))}\right)\tilde{z}^\Lambda$$

with

$$K(u,t)=\frac{P(\underline{\alpha}_1(u),\dots,\underline{\alpha}_{\underline{n}}(u))}{t}.$$

We write $\Lambda\in\mathfrak{a}_{\mathbb{Z}}$ as $\Lambda=\lambda+m\gamma$ with $\lambda\in\mathfrak{a}_{\mathbb{Z}}$ and $m\in\mathbb{Z}$. Then we have

$$p_\Lambda(u,t)=p_\lambda(u)\prod_{k=1}^{l}(t-\theta_k(u))^{m-\langle\theta_k,\lambda\rangle},$$

while $(\tilde{z})^\Lambda$ is simply z^λ. To simplify our notation, we introduce

$$P_{\underline{\mathfrak{A}}}(u)=P(\underline{\alpha}_1(u),\dots,\underline{\alpha}_{\underline{n}}(u)).$$

This is a polynomial on $\underline{\mathfrak{a}}_{\mathbb{C}}$ of degree d, but it can be considered as a polynomial on $\mathfrak{a}_{\mathbb{C}}$ as well.

Thus, after some simplifications, changing m to $m-1$ and using the absolute convergence of the series, we obtain that the toric residue (3.6) is equal to

$$\sum_{\lambda\in\mathfrak{a}_{\mathbb{Z}},m\in\mathbb{Z}}\ \sum_{F\in\mathcal{FL}(\mathfrak{A},\xi)}\nu(F)\operatorname*{Res}_F\left(\frac{P_{\underline{\mathfrak{A}}}(u)}{tp_\lambda(u)\prod_{k=1}^{l}(t-\theta_k(u))^{m-\langle\theta_k,\lambda\rangle}}\frac{1}{\prod_{i=1}^{\underline{n}}\underline{\alpha}_i(u)}\right)z^\lambda.$$

According to Proposition 3.4, for $\underline{\xi}\in\mathfrak{c}$ and $\xi=\underline{\xi}+bt$, for any $b\in\mathbb{R}$, in particular for $b=0$, we have

$$\mathcal{FL}(\mathfrak{A},\xi)=\{(\underline{F},\underline{\mathfrak{a}}^*);\ \underline{F}\in\mathcal{FL}(\underline{\mathfrak{A}},\underline{\xi})\}\tag{3.7}$$

Let us take $\Lambda=\lambda+m\gamma\in\mathfrak{a}_{\mathbb{Z}}$, fix $F\in\mathcal{FL}(\mathfrak{A},\xi)$, and denote by $a(F,\Lambda)$ the corresponding term in the above sum, omitting the sign $\nu(F)$. Thus

$$a(F,\Lambda)=\operatorname*{Res}_F\left(\frac{P_{\underline{\mathfrak{A}}}(u)}{tp_\lambda(u)\prod_{k=1}^{l}(t-\theta_k(u))^{m-\langle\theta_k,\lambda\rangle}}\frac{1}{\prod_{i=1}^{\underline{n}}\underline{\alpha}_i(u)}\right).$$

Then translating the relation (3.7) into the language of iterated residues, we obtain

$$a(F,\Lambda)=\operatorname*{Res}_{t=0}\frac{dt}{t}\operatorname*{Res}_{\underline{F}}\left(\prod_{k=1}^{l}(t-\theta_k(u))^{\langle\theta_k,\lambda\rangle-m}\frac{P_{\underline{\mathfrak{A}}}(u)}{p_\lambda(u)\prod_{i=1}^{\underline{n}}\underline{\alpha}_i(u)}\right).\tag{3.8}$$

Lemma 3.8. *Let $\Lambda=\lambda+m\gamma\in\mathfrak{a}_{\mathbb{Z}}$. Then $a(F,\Lambda)=0$ unless $m=0$.*

Proof. The proof will follow from degree considerations. Since $t=0$ is the last residue, in order to compute (3.8), we need to write

$$\prod_{k=1}^{l}(t-\theta_k(u))^{\langle\theta_k,\lambda\rangle-m}=t^{-lm+\langle\underline{\kappa},\lambda\rangle}\prod_{k=1}^{l}\left(1-\frac{\theta_k(u)}{t}\right)^{\langle\theta_k,\lambda\rangle-m}\tag{3.9}$$

and expand $(1-\theta_k(u)/t)^{\langle\theta_k,\lambda\rangle-m}$ in a power series involving positive powers of $(\theta_k(u)/t)$. We will track $\deg_t$, the t-degree, and $\deg_{\underline{\mathfrak{A}}}$, the degree of the rational function on $\underline{\mathfrak{a}}$. Clearly, after the expansion of the expression in parentheses in (3.8) we will need to retain the terms with

$$\deg_t = 0 \quad \text{and} \quad \deg_{\underline{\mathfrak{A}}} = -\underline{r}. \tag{3.10}$$

Now note that $\deg_{\underline{\mathfrak{A}}} P_{\underline{\mathfrak{A}}} = \underline{d}$, $\deg_{\underline{\mathfrak{A}}} p_\lambda = \langle\underline{\kappa},\lambda\rangle$, and the difference of degrees $\deg_t - \deg_{\underline{\mathfrak{A}}}$ of the expression (3.9) is equal to $\langle\underline{\kappa},\lambda\rangle - lm$. Hence we can conclude that the difference of degrees

$$[\deg_t - \deg_{\underline{\mathfrak{A}}}]\left(\prod_{k=1}^{l}(t-\theta_k(u))^{\langle\theta_k,\lambda\rangle-m}\frac{P_{\underline{\mathfrak{A}}}(u)}{p_\lambda(u)\prod_{i=1}^{\underline{n}}\underline{\alpha}_i(u)}\right)$$

of the expression in (3.8) is equal to $\underline{r}-lm$. Comparing this to (3.10) we see that the only possibility is $m=0$. $\square$

Now we are ready to finish the proof of Theorem 3.7. Taking into account Lemma 3.8, and switching back the order of summation, we have

$$\mathrm{TorRes}_{\mathfrak{f}(\tilde{z})} e_\gamma P(z_1 e_{\mu_1},\dots,z_{\underline{n}}e_{\mu_{\underline{n}}}) = \sum_{\lambda\in\underline{\mathfrak{a}}_{\mathbb{Z}}}$$

$$\sum_{\underline{F}\in\mathcal{FL}(\underline{\mathfrak{A}},\underline{\xi})}\nu(\underline{F})\operatorname*{Res}_{t=0}\frac{dt}{t}\operatorname*{Res}_{\underline{F}}\left(\prod_{k=1}^{l}(t-\theta_k(u))^{\langle\theta_k,\lambda\rangle}\frac{P_{\underline{\mathfrak{A}}}(u)}{p_\lambda(u)\prod_{i=1}^{\underline{n}}\underline{\alpha}_i(u)}\right)z^\lambda.$$

We observe that since $\underline{\xi}$ is in chamber $\mathfrak{c}$ containing $\underline{\kappa}=\kappa(\underline{\mathfrak{A}})$ in its closure, according to Proposition 2.16 and Theorem 2.14, we can replace the sum of iterated residues by a Jeffrey-Kirwan residue, and we arrive at the expression

$$\sum_{\lambda\in\underline{\mathfrak{a}}_{\mathbb{Z}}}\operatorname*{Res}_{t=0}\frac{dt}{t}\mathrm{JK}_{\mathfrak{c}}\left(\prod_{k=1}^{l}(t-\theta_k(u))^{\langle\theta_k,\lambda\rangle}\frac{P_{\underline{\mathfrak{A}}}(u)}{p_\lambda(u)\prod_{i=1}^{\underline{n}}\underline{\alpha}_i(u)}\right)z^\lambda.$$

for the toric residue.

Next, using Lemma 2.12, we can restrict the summation on the right hand side to the integral vectors λ in $\bar{\mathfrak{c}}^\perp$. Now we use our assumption that the vectors θ_k are in $\bar{\mathfrak{c}}$, which implies that $\langle\theta_k,\lambda\rangle \geq 0$ for every $\lambda\in\bar{\mathfrak{c}}^\perp$ and $k=1,\dots,l$. This means that all the exponents in the product

$$\prod_{k=1}^{l}(t-\theta_k(u))^{\langle\theta_k,\lambda\rangle}$$

are nonnegative, which immediately implies that only the term

$$\prod_{k=1}^{l}\theta_k(u)^{\langle\theta_k,\lambda\rangle},$$

can contribute to the residue with respect to t. Thus we are left with the equality

$$\mathrm{TorRes}_{\mathbf{f}(\tilde{z})} e_\gamma P(z_1 e_{\mu_1}, \dots, z_{\underline{n}} e_{\mu_{\underline{n}}}) = \sum_{\lambda \in \underline{\mathfrak{a}}_{\mathbb{Z}} \cap \bar{\mathfrak{c}}^\perp} \mathrm{JK}_{\mathfrak{c}} \left(\prod_{k=1}^{l} \theta_k(u)^{\langle \theta_k, \lambda \rangle} \frac{P_{\underline{\mathfrak{A}}}(u)}{p_\lambda(u) \prod_{i=1}^{\underline{n}} \underline{\alpha}_i(u)} \right) z^\lambda.$$

This completes the proof.

□

4. Appendix

We start again with the setting of §1. Let $z = (z_1, z_2, \dots, z_n)$ be a vector in $\mathbb{C}^n$, and consider the element

$$f_z = \sum_{i=1}^{n} z_i e_{\mu_i} \in S^1(C).$$

For each $a \in V^*$, we consider the derivative

$$f_{z,a} = \sum_{i=1}^{n} z_i \langle a, \mu_i \rangle e_{\mu_i}$$

of the function f_z in the direction of the vector $a \in V^*$.

Choose a basis $[a_0, a_1, \dots, a_d]$ of V^*, where $a_0 = g$, the grading vector. Our goal is to describe a suitable set Z of values of the vector parameter z for which the sections $f_{z,j} = f_{z,a_j}$ do satisfy the conditions (1.4) and (1.5). The first condition is

$$\{x \in \mathrm{Tvar}_g(C);\ f_{z,a_j}(x) = 0,\ j = 0, \dots, d\} = \emptyset \tag{4.1}$$

when $z \in Z$. Note that this condition does not depend of the choice of the basis $[g, a_1, \dots, a_d]$. In this appendix, we recall an argument we learned from Alicia Dickenstein to determine such a suitable set Z of values of the vector parameter z.

Recall the notations of §1.3. Let $\mathfrak{g} = \mathbb{R}^n$. We denote by ω_i the canonical basis of $\mathbb{R}^n$ so that

$$\mathfrak{g} = \oplus_{i=1}^{n} \mathbb{R}\omega_i.$$

We introduce the corresponding lattice

$$\mathfrak{g}_{\mathbb{Z}} = \oplus_{i=1}^{n} \mathbb{Z}\omega_i.$$

The dual basis $\omega^i \in \mathfrak{g}^*$ is the set of coordinates on $\mathfrak{g}$. Consider the surjection $\mathfrak{g} \to V$ sending ω_i to μ_i. Our grading vector $g \in V^*$ takes value 1 on each μ_i; let $W \subset \mathfrak{g}$ be the kernel of this map. We then obtain an exact sequence

$$0 \to W \to \mathfrak{g} \to V \to 0.$$

We denote by $W_{\mathbb{Z}}$ the intersection of the lattice $\mathfrak{g}_{\mathbb{Z}}$ with W. We denote by ν_i the restriction to W of the coordinate ω^i. An element $w \in W$ is written in

the basis ω_i as $w = \sum_{i=1}^n \nu_i(w)\omega_i$. By definition, for all $w \in W$ and $a \in V^*$, we have the relation

$$\sum_{i=1}^n \nu_i(w)\mu_i(a) = 0. \tag{4.2}$$

Note that if $m = \sum_{i=1}^n m_i\omega_i \in W$ we have $\sum_{i=1}^n m_i = 0$, as follows from the relation $\sum_{i=1}^n m_i\mu_i(g) = 0$.

We start our analysis of the common set of zeroes of the functions f_{z,a_j}.

Let $x \in \mathrm{Spec}(S(C))$ be an element of the affine variety $\mathrm{Aff}(C)$. Recall the notation x_i for $x(e_{\mu_i})$.

Lemma 4.1. *If $x \in \mathrm{Aff}(C)$ is a common zero of the functions $f_{z,a_j}, j = 0, \dots, d$, we have the relation*

$$\sum_{i=1}^n z_i x_i \omega_i \in W_{\mathbb{C}}.$$

Proof. Considering the value of x on $f_{z,a_j} \in S^1(C)$, we obtain the set of equations $\sum_{i=1}^n z_i \langle a_j, \mu_i\rangle x_i = 0$ where a_j runs through a basis of V^*. Thus $\sum_{i=1}^n z_i x_i \mu_i = 0$, which exactly says that $\sum_{i=1}^n z_i x_i \omega_i$ belongs to $W_{\mathbb{C}}$.

□

Now we analyze conditions on the vector parameter z which guarantee that the equation $\sum_{i=1}^n z_i x_i \omega_i \in W_{\mathbb{C}}$ together with the fact that x belongs to $\mathrm{Aff}(C)$ implies that $x = 0$. We need to recall the equations of $\mathrm{Aff}(C)$.

For every element $m = \sum_{i=1}^n m_i\omega_i \in W_{\mathbb{Z}}$, we have $\sum_{i=1}^n m_i\mu_i = 0$, which, in turn, implies the following relation in $S(C)$:

$$\prod_{\{i;\, m_i>0\}} (e_{\mu_i})^{m_i} = \prod_{\{i;\, m_i<0\}} (e_{\mu_i})^{-m_i}. \tag{4.3}$$

This gives us the set of binomial homogeneous equations on $x \in \mathrm{Aff}(C)$:

$$\prod_{\{i;\, m_i>0\}} x_i^{m_i} = \prod_{\{i;\, m_i<0\}} x_i^{-m_i} \tag{4.4}$$

We identify $\mathfrak{g}_{\mathbb{C}}$ with $\mathbb{C}^n$ with the help of the basis ω_i. For every $m = \sum_{i=1}^n m_i\omega_i \in \mathfrak{g}_{\mathbb{Z}}$, we introduce the rational function $p_m(z) = \prod_{i=1}^n z_i^{m_i}$ on $\mathfrak{g}_{\mathbb{C}}$, which is a Laurent monomial of homogeneous degree $\sum_{i=1}^n m_i$. It is well defined on $(\mathbb{C}^*)^n$. Note that the rational functions $p_m(z)$ have the multiplicative property

$$p_{m_1}p_{m_2} = p_{m_1+m_2}$$

for $m_1, m_2 \in \mathfrak{g}_{\mathbb{Z}}$.

Equation (4.4) shows that if $x \in \mathrm{Aff}(C)$ is such that x_i is non zero for all i, then for all $m \in W_{\mathbb{Z}}$, the number $p_m(x_1, x_2, \dots, x_n) = \prod_{i=1}^n x_i^{m_i}$ is equal to 1. Our strategy is to show that there exists a non-empty finite set S in $W_{\mathbb{Z}}$ and constants $c_m \neq 0$ such that the relation $\sum_{m\in S} c_m p_m = 0$ holds

identically on $W_{\mathbb{C}}$. If we evaluate $\sum_{m\in S} c_m p_m$ on the element $\sum_{i=1}^n z_i x_i \omega_i$ of $W_{\mathbb{C}}$, we obtain the relation $\sum_{m\in S} c_m p_m(z_1, z_2, \dots, z_n) = 0$. In particular, if our vector $z \in \mathbb{C}^n$ is chosen outside the hypersurface $\sum_{m\in S} c_m p_m = 0$, the equation $\sum_{i=1}^n z_i x_i \omega_i \in W_{\mathbb{C}}$ cannot hold. We will need to refine this argument to obtain also a contradiction, when some of the x_i vanishes.

For $x \in \mathrm{Aff}(C)$, let us analyze the set $I(x) \subset \{1, 2, \dots, n\}$ of elements i with $x_i \neq 0$.

Lemma 4.2. *Let $x \in \mathrm{Aff}(C)$. If $x \neq 0$, then there exists a face Φ of the cone C such that $x_i \neq 0$ if and only if $\mu_i \in \Phi$.*

Proof. If $x \neq 0$, the set $I := I(x)$ is not empty. Consider the cone $C(I)$ generated by the elements $\{\mu_i; i \in I\}$. Let us prove that this cone is a face of the polyhedral cone C. Consider a minimal face Φ among the faces containing $C(I)$. Then $C(I)$ contains an interior point μ of Φ. We can write $N\mu = \sum_{i\in I} n_i \mu_i$ with N a positive integer and $n_i \geq 0$. The relation $e_{N\mu} = \prod_{i\in I} e_{n_i\mu_i}$ insures that $x(e_\mu) \neq 0$. But, as μ is an interior point of Φ, there is also a relation $L\mu = \sum_{k,\mu_k\in\Phi} n_k \mu_k$ where L and n_k are strictly positive integers. The relation $x(e_\mu)^L = \prod_{k,\mu_k\in\Phi} x(e_{\mu_k})^{n_k}$ implies that $x_k \neq 0$ for all k such that $\mu_k \in \Phi$. Thus $C(I) = \Phi$.

□

Let Φ be a face of the cone C. Let $I(\Phi)$ be the subset of $\{1, 2, \dots, n\}$ such that $\mu_i \in \Phi$. Let $\mathfrak{g}_\Phi = \oplus_{i\in I(\Phi)} \mathbb{R}\omega_i$ and $W_\Phi = W \cap \mathfrak{g}_\Phi$.

Assume W_Φ is not equal to $\{0\}$. This means that there exists a non trivial relation between the elements μ_i of $\mathfrak{M}$ belonging to the face Φ. For each $i \in I(\Phi)$, we still denote by ν_i the restriction to $W_\Phi \subset W \subset \mathfrak{g}$ of the coordinate ω^i and we denote by $U_\Phi := \{u \in W_{\Phi,\mathbb{C}}, \prod_{i\in I(\Phi)} \nu_i(u) \neq 0\}$. For $m = \sum_{i\in I(\Phi)} m_i \omega_i$ in $\mathfrak{g}_{\Phi,\mathbb{Z}}$, the restriction of the function $p_m(z) = \prod_{i\in I(\Phi)} z_i^{m_i}$ to $W_{\Phi,\mathbb{C}}$ is well defined on the open set U_Φ of $W_{\Phi,\mathbb{C}}$.

Lemma 4.3. *Let Φ be a face of C. If $W_\Phi \neq \{0\}$, then there exists a non-empty finite set $S_\Phi \subset W_{\Phi,\mathbb{Z}}$ and constants $c_m \in \mathbb{C}^*$ such that the relation $\sum_{m\in S_\Phi} c_m p_m = 0$ holds identically on U_Φ.*

Proof. Let us consider a basis $\{w_1, w_2, \dots, w_{r_\Phi}\}$ of $W_{\Phi,\mathbb{Z}} \subset \mathfrak{g}_{\mathbb{Z}}$. Here r_Φ is the dimension of W_Φ. We denote simply by p_j the rational function p_{w_j} on $\mathbb{C}^n$. Each of the function p_j is homogeneous of degree 0. The rational map $p = (p_1, p_2, \dots, p_{r_\Phi})$ defines by restriction a map from U_Φ to $(\mathbb{C}^*)^{r_\Phi}$ which is homogeneous of degree 0. Thus its image is of dimension strictly less than r_Φ. This implies that there is a non zero polynomial Q such that $Q(p_1(u), \dots, p_{r_\Phi}(u))$ is identically 0 on U_Φ. Using the relation $p_w p_{w'} = p_{w+w'}$, for $w, w' \in W_{\mathbb{Z}}$, the relation $Q(p_1(u), \dots, p_{r_\Phi}(u))$ is equivalent to a relation of the form given in the lemma. □

Definition 4.1. For each face Φ of the cone C satisfying $W_\Phi \neq \{0\}$, we choose a non-empty finite set $S_\Phi \subset W_{\Phi,\mathbb{Z}}$ and non zero constants $c_m, m \in S_\Phi$,

such that the rational function $R_\Phi = \sum_{m\in S_\Phi} c_m p_m$ is identically equal to 0 on U_Φ. We define

$$R = \prod R_\Phi,$$

where the product runs over the faces Φ of C with $W_\Phi \neq \{0\}$.

Definition 4.2. Let σ be a subset of $\{1, 2, \dots, n\}$. Denote by

$$Z_\sigma := \{z = (z_1, z_2, \dots, z_n) \in (\mathbb{C}^*)^n \,;\, z_i = 1, i \in \sigma\}.$$

Lemma 4.4. *Let σ be a subset of $\{1, 2, \dots, n\}$ such that the set $\{\mu_i, i \in \sigma\}$ is a set of linearly independent vectors of V. Then the function R does not vanish identically on Z_σ.*

Proof. The function R is of the form $\sum_{m\in W_\mathbb{Z}} k_m p_m$, where k_m is non zero for a finite non-empty set of $m \in W_\mathbb{Z}$. Let

$$Y_\sigma = \{y = (y_1, y_2, \dots, y_n) \in \mathbb{C}^n \,;\, y_i = 0, i \in \sigma\}.$$

Denote by e^y the element $(e^{y_1}, \dots, e^{y_n})$ of Z_σ. For $m \in W_\mathbb{Z}$, the function p_m is written as the exponential function

$$p_m(e^y) = e^{\sum_{k\notin\sigma}^n y_k \nu_k(m)}.$$

By Gale duality, if the vectors μ_i for $i \in \sigma$ are linearly independent, then the linear forms ν_k for $k \notin \sigma$ span W^*. For $m_1 \neq m_2 \in W_\mathbb{Z}$, the two linear functions $y \mapsto \sum_{k\notin\sigma} y_k \nu_k(m_1)$ and $\sum_{k\in\notin\sigma} y_k \nu_k(m_2)$ are not identically equal on Y_σ. Otherwise, we would obtain $\sum_{k\notin\sigma} y_k \nu_k(m_1 - m_2) = 0$. This implies $m_1 = m_2$ as $\sum_{k\notin\sigma} y_k \nu_k$ ranges through W^*, when y ranges through Y_σ. Thus the relation $\sum_{m\in W_\mathbb{Z}} k_m p_m(e^y) = 0$ between different exponentials cannot hold identically on Y_σ. □

In particular, the set of $z \in Z_\sigma$ where $R(z) \neq 0$ is an open dense set of Z_σ. As R is the product of the functions R_Φ over the faces Φ such that $W_\Phi \neq \{0\}$, it follows that the condition $R(z) \neq 0$ implies that $R_\Phi(z) \neq 0$ for any face Φ of C such that $W_\Phi \neq \{0\}$.

Finally, we obtain a condition which ensures that the sections $\{f_{z,a_j}, j = 0, \dots, d\}$ do not have common zeroes on $\mathrm{Tvar}_g(C)$.

Proposition 4.5. *Let σ be a subset of $\{1, 2, \dots, n\}$ such that the set $\{\mu_i, i \in \sigma\}$ is a set of linearly independent vectors. Then for z varying in the open dense set Z of Z_σ where $R(z) \neq 0$, the $d + 1$ sections $f_{z,a_j}, j = 0, \dots, d$ do not have a common zero in $\mathrm{Tvar}_g(C)$.*

Proof. Consider a common zero x of the functions f_{z,a_j} on $\mathrm{Aff}(C)$. We need to prove that $x_i = 0$ for all $i \in \{1, 2, \dots, n\}$. If not, there is a face Φ of C, with $x_i \neq 0$ if and only $\mu_i \in \Phi$. The relation $\sum_{i=1}^n z_i x_i \mu_i = 0$ implies that $W_\Phi \neq \{0\}$ for this face Φ. The element $\sum_{i=1}^n z_i x_i \omega_i$ is in U_Φ.

Consider the rational function $R_\Phi = \sum_{m\in S_\Phi} c_m p_m$ with $S_\Phi \subset W_{\Phi,\mathbb{Z}}$. For $m \in W_{\Phi,\mathbb{Z}}$, the value $p_m(x_1, x_2, \dots, x_n)$ is well defined and equal to 1. Thus the value of the function $R_\Phi = \sum_{m\in S_\Phi} c_m p_m$ on the element $\sum_{i=1}^n z_i x_i \omega_i$ of

U_Φ is simply equal to $R_\Phi(z) = \sum_{m \in S_F} c_m p_m(z) \neq 0$ by our assumption on z. This contradicts the fact that R_Φ vanishes identically on U_Φ.

□

Thus, when z is generic in Z_σ, the $d+1$ sections $f_{z,a_j}, j = 0, \ldots, d$ satisfy the condition (4.1) which ensures the existence of the toric residue.

Recall that we have chosen a vector $\gamma \in V_{\mathbb{Z}}$ in the interior of the cone C and that $\mathfrak{t}^* \subset V^*$ is the orthogonal of γ. The space $\mathfrak{t}^*$ is a rational subspace of dimension d of V^*.

Proposition 4.6. *Let $[a_1, a_2, \ldots, a_d]$ be a basis of $\mathfrak{t}^*$. Let σ be a subset of $\{1, 2, \ldots, n\}$ such that the set $\{\mu_i, i \in \sigma\}$ is a set of linearly independent vectors. Then for z varying in the open dense subset Z of Z_σ where $R(z) \neq 0$, the common zeros x of the sections f_{z,a_j} for $j = 1, \ldots, d$ lie in the torus $\mathrm{T}_{\mathbb{C}}(H^*)$.*

Proof. As before, let $x \in \mathrm{Aff}(C) \setminus \{0\}$. If x is a zero of the functions $f_{z,a}$ for all $a \in \mathfrak{t}^*$, we want to prove that $x_i \neq 0$ for all i.

From the equations $f_{z,a}(x) = 0$, we obtain that

$$\sum_{i=1}^{n} z_i \langle a, \mu_i \rangle x_i = 0$$

for all $a \in V^*$ orthogonal to γ. Thus there exists $t \in \mathbb{C}$ such that

$$\sum_{i=1}^{n} z_i x_i \mu_i = t\gamma.$$

If $t = 0$, then the point x is a zero of all sections $f_{z,a}$ for all $a \in V^*$ and the condition $R(z) \neq 0$ implies $x = 0$. So we may assume $t \neq 0$, and γ belongs to the space generated by the μ_i with $x_i \neq 0$. In particular there is a relation $N\gamma = \sum_{i, x_i \neq 0} n_i \mu_i$ with $N > 0$ and $n_i \in \mathbb{Z}$. This implies that $x(e_\gamma) \neq 0$.

We have seen that there exists a face Φ of C such that $x_i \neq 0$ if an only $\mu_i \in \Phi$. As γ is an interior point of C, this face is necessarily the cone C itself. This concludes the proof of the proposition. □

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